On the Commuting variety of a reductive Lie algebra and other related varieties.

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ON THE COMMUTING VARIETY OF A REDUCTIVE LIE ALGEBRA AND OTHER RELATED VARIETIES.

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ABSTRACT. The nilpotent cone of a reductive Lie algebra has a desingularization given by the cotangent bundle of the flag variety. Analogously, the nullcone of a cartesian power of the algebra has a desingularization given by a vector bundle over the flag variety. As for the nullcone, the subvariety of elements whose components are in a same Borel subalgebra, has a desingularization given by a vector bundle over the flag variety. In this note, we study geometrical properties of these varieties. For the study of the commuting variety, the analogous variety to the flag variety is the closure in the Grassmannian of the set of Cartan subalgebras. So some properties of this variety are given. In particular, it is smooth in codimension 1. We introduce the generalized isospectral commuting varieties and give some properties. Furthermore, desingularizations of these varieties are given by fiber bundles over a desingularization of the closure in the grassmannian of the set of Cartan subalgebras contained in a given Borel subalgebra.

CONTENTS

1. INTRODUCTION

In this note, the base field $\Bbbk$ is algebraically closed of characteristic 0, $\mathfrak{g}$ is a reductive Lie algebra of finite dimension, ℓ is its rank, $\dim \mathfrak{g} = \ell + 2n$ and G is its adjoint group. As usual, $\mathfrak{b}$ denotes a Borel subalgebra of $\mathfrak{g}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, contained in $\mathfrak{b}$, and B the normalizer of $\mathfrak{b}$ in G.

1.1. **Main results.** Let $\mathcal{B}^{(k)}$ be the subset of elements $(x_1, \ldots, x_k)$ of $\mathfrak{g}^k$ such that $x_1, \ldots, x_k$ are in a same Borel subalgebra of $\mathfrak{g}$. This subset of $\mathfrak{g}^k$ is closed and contains two interesting subsets: the nullcone of $\mathfrak{g}^k$ denoted by $\mathcal{N}^{(k)}$ and the generalized commuting variety of $\mathfrak{g}$ that is the closure in $\mathfrak{g}^k$ of the subset of elements $(x_1, \ldots, x_k)$ such that $x_1, \ldots, x_k$ are in a same Cartan subalgebra of $\mathfrak{g}$. We denote it by $\mathcal{C}^{(k)}$. According to [Mu65, Ch.2, §1, Theorem], for $(x_1, \ldots, x_k)$ in $\mathcal{B}^{(k)}$, $(x_1, \ldots, x_k)$ is in $\mathcal{N}^{(k)}$ if and only if $x_1, \ldots, x_k$ are nilpotent. According to a Richardson Theorem [Ri79], $\mathcal{C}^{(2)}$ is the commuting variety of $\mathfrak{g}$.

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There is a natural projective morphism $G\times_B \mathfrak{b}^k \longrightarrow \mathcal{B}^{(k)}$. For $k = 1$, this morphism is not birational but for $k \geq 2$, it is birational (see Lemma 2.2 and Lemma 2.4). Furthermore, denoting by $\mathcal{X}$ the subvariety of elements (x, y) of $\mathfrak{g}\times\mathfrak{h}$ such that y is in the closure of the orbit of x under G, the morphism

$$G\times\mathfrak{b} \longrightarrow \mathcal{X}, \qquad (g, x) \longmapsto (g.x, \overline{x})$$

with $\overline{x}$ the projection of x onto $\mathfrak{h}$ defines through the quotient a projective and birational morphism $G\times_B \mathfrak{b} \longrightarrow \mathcal{X}$ and $\mathfrak{g}$ is the categorical quotient of $\mathcal{X}$ under the action of $W(\mathcal{R})$ on the factor $\mathfrak{h}$, with $W(\mathcal{R})$ the Weyl group of $\mathfrak{g}$. For $k \geq 2$, the inverse image of $\mathcal{B}^{(k)}$ by the canonical projection from $\mathcal{X}^k$ to $\mathfrak{g}^k$ is not irreducible but the canonical action of $W(\mathcal{R})^k$ on $\mathcal{X}^k$ induces a simply transitive action on the set of its irreducible components. Setting:

$$\mathcal{B}_{\mathrm{x}}^{(k)} := \{((g(x_1), \overline{x_1}), \dots, (g(x_k), \overline{x_k})) \mid (g, x_1, \dots, x_k) \in G\times\mathfrak{b}^k\},$$

$\mathcal{B}_{\mathrm{x}}^{(k)}$ is an irreducible component of the inverse image of $\mathcal{B}^{(k)}$ in $\mathcal{X}^k$ (see Corollary 2.8) and we have a commutative diagram

$$\begin{tikzcd} G\times_B \mathfrak{b}^k \arrow[rr] \arrow[dr, "\gamma"'] & & \mathcal{B}_{\mathrm{x}}^{(k)} \arrow[dl, "\eta"] \\ & \mathcal{B}^{(k)} & \end{tikzcd}$$

with η the restriction to $\mathcal{B}_{\mathrm{x}}^{(k)}$ of the canonical projection ϖ from $\mathcal{X}^k$ to $\mathfrak{g}^k$. The first main theorem of this note is the following theorem:

Theorem 1.1. (i) *The variety $\mathcal{N}^{(k)}$ is normal if and only if so is $\mathcal{B}_{\mathrm{x}}^{(k)}$.*

(ii) *The variety $\mathcal{N}^{(k)}$ is Cohen-Macaulay if and only if so is $\mathcal{B}_{\mathrm{x}}^{(k)}$.*

(iii) *The variety $\mathcal{N}^{(k)}$ has rational singularities if and only if it is Cohen-Macaulay.*

(iv) *The variety $\mathcal{B}_{\mathrm{x}}^{(k)}$ has rational singularities if and only if it is Cohen-Macaulay.*

(v) *The algebra $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]$ is a free extension of $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]^G$ which identifies with* $\mathrm{S}(\mathfrak{h}^k)$.

(vi) *The algebra $\Bbbk[\mathcal{B}^{(k)}]^G$ identifies with* $\mathrm{S}(\mathfrak{h}^k)^{W(\mathcal{R})}$ *with respect to the diagonal action of $W(\mathcal{R})$ in $\mathfrak{h}^k$.*

(vii) *The ideal $\Bbbk[\mathcal{B}^{(k)}]\Bbbk[\mathcal{B}^{(k)}]_+^G$ of $\Bbbk[\mathcal{B}^{(k)}]$ is strictly contained in the ideal of definition of $\mathcal{N}^{(k)}$ in $\Bbbk[\mathcal{B}^{(k)}]$.*

According to K. Vilonen and T. Xue [VX15], $\mathcal{N}^{(k)}$ and $\mathcal{B}_{\mathrm{x}}^{(k)}$ are not normal in general. In the study of the generalized commuting variety, the closure in $\mathrm{Gr}_\ell(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under the action of G plays an analogous role to the flag variety. Denoting by X the closure in $\mathrm{Gr}_\ell(\mathfrak{b})$ of the orbit of $\mathfrak{h}$ under B, $G.X$ is the closure of the orbit of $G.\mathfrak{h}$ and we have the following second main result:

Theorem 1.2. *Let X' be the set of centralizers of regular elements of $\mathfrak{b}$ whose semisimple components is regular or subregular.*

(i) *All element of X is a commutative algebraic subalgebra of $\mathfrak{g}$.*

(ii) *For x in $\mathfrak{g}$, the set of elements of $G.X$ containing x has dimension at most* $\dim\mathfrak{g}^x - \ell$.

(iii) *The sets $X\setminus B.\mathfrak{h}$ and $G.X\setminus G.\mathfrak{h}$ are equidimensional of dimension $n-1$ and $2n-1$ respectively.*

(iv) *The sets X' and $G.X'$ are smooth big open subsets of X and $G.X$ respectively.*

This is a main result with respect to the generalized commuting varieties as it will be shown in the next two notes. We recall that an element of $\mathfrak{g}$ is subregular if its centralizer in $\mathfrak{g}$ has dimension

$\ell+2$. Let $\mathfrak{X}_{0,k}$ be the closure in $\mathfrak{b}^k$ of $B.\mathfrak{h}^k$ and let Γ be a desingularization of X in the category of B-varieties. Let $\mathcal{E}_0$ be the tautological bundle over X and set:

$$\mathcal{E}_{\mathrm{s}} := \mathcal{E}_0 \times_X \Gamma, \qquad \mathcal{E}_{\mathrm{s}}^{(k)} := \underbrace{\mathcal{E}_{\mathrm{s}} \times_\Gamma \cdots \times_\Gamma \mathcal{E}_{\mathrm{s}}}_{k \text{ factors}}.$$

Then $\mathcal{E}_{\mathrm{s}}^{(k)}$ is a desingularization of $\mathfrak{X}_{0,k}$. Set: $\mathcal{C}_{\mathrm{x}}^{(k)} := \eta^{-1}(\mathcal{C}^{(k)})$. The following theorem is the third main result of this note:

Theorem 1.3. *The variety $\mathcal{C}_{\mathrm{x}}^{(k)}$ is irreducible and $G \times_B \mathcal{E}_{\mathrm{s}}^{(k)}$ is a desingularization of $\mathcal{C}_{\mathrm{x}}^{(k)}$.*

It will be proved in a next note that the normalizations of $\mathfrak{X}_{0,k}$, $\mathcal{C}^{(k)}$ and $\mathcal{C}_{\mathrm{x}}^{(k)}$ are Gorenstein with rational singularities. As a matter of fact, as a consequence, $\mathfrak{X}_{0,k}$ is normal.

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1.2. **Notations.** • An algebraic variety is a reduced scheme over $\Bbbk$ of finite type.

• For V a vector space, its dual is denoted by V^* and the augmentation ideal of its symmetric algebra $\mathrm{S}(V)$ is denoted by $\mathrm{S}_+(V)$. For A a graded algebra over $\mathbb{N}$, A_+ is the ideal generated by the homogeneous elements of positive degree.

• All topological terms refer to the Zariski topology. If Y is a subset of a topological space X, denote by $\overline{Y}$ the closure of Y in X. For Y an open subset of the algebraic variety X, Y is called *a big open subset* if the codimension of $X \setminus Y$ in X is at least 2. For Y a closed subset of an algebraic variety X, its dimension is the biggest dimension of its irreducible components and its codimension in X is the smallest codimension in X of its irreducible components. For X an algebraic variety, $\mathcal{O}_X$ is its structural sheaf, $\Bbbk[X]$ is the algebra of regular functions on X and $\Bbbk(X)$ is the field of rational functions on X when X is irreducible.

• For X an algebraic variety and for $\mathcal{M}$ a sheaf on X, $\Gamma(V,\mathcal{M})$ is the space of local sections of $\mathcal{M}$ over the open subset V of X. For i a nonnegative integer, $\mathrm{H}^i(X,\mathcal{M})$ is the i-th group of cohomology of $\mathcal{M}$. For example, $\mathrm{H}^0(X,\mathcal{M}) = \Gamma(X,\mathcal{M})$.

Lemma 1.4. [EGAII, Corollaire 5.4.3] *Let X be an irreducible affine algebraic variety and let Y be a desingularization of X. Then $\mathrm{H}^0(Y,\mathcal{O}_Y)$ is the integral closure of $\Bbbk[X]$ in its fraction field.*

• For K a group and for E a set with a group action of K, E^K is the set of invariant elements of E under K. The following lemma is straightforward and will be used in the proof of Corollary 2.23.

Lemma 1.5. *Let A be an algebra generated by the subalgebras A_1 and A_2. Let K be a group acting on A_2. Suppose that the following conditions are verified:*

(1) *$A_1 \cap A_2$ is contained in A_2^K,*
(2) *A is a free A_2-module having a basis contained in A_1,*
(3) *A_1 is a free $A_1 \cap A_2$-module having the same basis.*

Then there exists a unique group action of K on the algebra A extending the action of K on A_2 and fixing all the elements of A_1. Moreover, if $A_1 \cap A_2 = A_2^K$ then $A^K = A_1$.

• For E a finite set, its cardinality is denoted by $|E|$. For E a vector space and for $x = (x_1,\dots,x_k)$ in E^k, E_x is the subspace of E generated by $x_1,\dots,x_k$. Moreover, there is a canonical action of

$\mathrm{GL}_k(\Bbbk)$ in E^k given by:

$$(a_{i,j}, 1 \leq i, j \leq k).(x_1, \ldots, x_k) := (\sum_{j=1}^{k} a_{i,j}x_j, i = 1, \ldots, k)$$

In particular, the diagonal action of G in $\mathfrak{g}^k$ commutes with the action of $\mathrm{GL}_k(\Bbbk)$.

• For $\mathfrak{a}$ reductive Lie algebra, its rank is denoted by $\mathrm{rk}_\mathfrak{a}$ and the dimension of its Borel subalgebras is denoted by $\mathrm{b}_\mathfrak{a}$. In particular, $\dim \mathfrak{a} = 2\mathrm{b}_\mathfrak{a} - \mathrm{rk}_\mathfrak{a}$.

• If E is a subset of a vector space V, denote by $\mathrm{span}(E)$ the vector subspace of V generated by E. The grassmanian of all d-dimensional subspaces of V is denoted by $\mathrm{Gr}_d(V)$. By definition, a *cone* of V is a subset of V invariant under the natural action of $\Bbbk^* := \Bbbk \setminus \{0\}$ and a *multicone* of V^k is a subset of V^k invariant under the natural action of $(\Bbbk^*)^k$ on V^k.

Lemma 1.6. *Let X be an open cone of V and let S be a closed multicone of $X \times V^{k-1}$. Denoting by S' the image of S by the first projection, $S' \times \{0\} = S \cap (X \times \{0\})$. In particular, S' is closed in X.*

Proof. For x in X, x is in S' if and only if for some $(v_2, \ldots, v_k)$ in V^{k-1}, $(x, tv_2, \ldots, tv_k)$ is in S for all t in $\Bbbk$ since S is a closed multicone of $X \times V^{k-1}$, whence the lemma. □

• The dual $\mathfrak{g}^*$ of $\mathfrak{g}$ identifies with $\mathfrak{g}$ by a given non degenerate, invariant, symmetric bilinear form $\langle .,. \rangle$ on $\mathfrak{g} \times \mathfrak{g}$ extending the Killing form of $[\mathfrak{g}, \mathfrak{g}]$.

• Let $\mathcal{R}$ be the root system of $\mathfrak{h}$ in $\mathfrak{g}$ and $\mathcal{R}_+$ the positive root system of $\mathcal{R}$ defined by $\mathfrak{b}$. The Weyl group of $\mathcal{R}$ is denoted by $W(\mathcal{R})$ and the basis of $\mathcal{R}_+$ is denoted by Π. The neutral elements of G and $W(\mathcal{R})$ are denoted by $1_\mathfrak{g}$ and $1_\mathfrak{h}$ respectively. For α in $\mathcal{R}$, the corresponding root subspace is denoted by $\mathfrak{g}^\alpha$ and a generator x_α of $\mathfrak{g}^\alpha$ is chosen so that $\langle x_\alpha, x_{-\alpha} \rangle = 1$ for all α in $\mathcal{R}$.

• The normalizers of $\mathfrak{b}$ and $\mathfrak{h}$ in G are denoted by B and $N_G(\mathfrak{h})$ respectively. For x in $\mathfrak{b}$, $\overline{x}$ is the element of $\mathfrak{h}$ such that $x - \overline{x}$ is in the nilpotent radical $\mathfrak{u}$ of $\mathfrak{b}$.

• For X an algebraic B-variety, denote by $G \times_B X$ the quotient of $G \times X$ under the right action of B given by $(g, x).b := (gb, b^{-1}.x)$. More generally, for k positive integer and for X an algebraic B^k-variety, denote by $G^k \times_{B^k} X$ the quotient of $G^k \times X$ under the right action of B^k given by $(g, x).b := (gb, b^{-1}.x)$ with g and b in G^k and B^k respectively.

Lemma 1.7. *Let P and Q be parabolic subgroups of G such that P is contained in Q. Let X be a Q-variety and let Y be a closed subset of X, invariant under P. Then $Q.Y$ is a closed subset of X. Moreover, the canonical map from $Q \times_P Y$ to $Q.Y$ is a projective morphism.*

Proof. Since P and Q are parabolic subgroups of G and since P is contained in Q, Q/P is a projective variety. Denote by $Q \times_P X$ and $Q \times_P Y$ the quotients of $Q \times X$ and $Q \times Y$ under the right action of P given by $(g, x).p := (gp, p^{-1}.x)$. Let $g \mapsto \overline{g}$ be the quotient map from Q to Q/P. Since X is a Q-variety, the map

$$Q \times X \longrightarrow Q/P \times X\ , \qquad (g, x) \longmapsto (\overline{g}, g.x)$$

defines through the quotient an isomorphism from $Q \times_P X$ to $Q/P \times X$. Since Y is a P-invariant closed subset of X, $Q \times_P Y$ is a closed subset of $Q \times_P X$ and its image by the above isomorphism is closed. Hence $Q.Y$ is a closed subset of X since Q/P is a projective variety. From the commutative diagram:

$$\begin{tikzcd} Q\times_P Y \arrow[r] \arrow[dr] & Q/P\times Q.Y \arrow[d] \\ & Q.Y \end{tikzcd}$$

we deduce that the map $Q\times_P Y \to Q.Y$ is a projective morphism. □

• For $k\geq 1$ and for the diagonal action of B in $\mathfrak{b}^k$, $\mathfrak{b}^k$ is a B-variety. The image of $(g,x_1,\dots,x_k)$ in $G\times\mathfrak{b}^k$ in $G\times_B\mathfrak{b}^k$ is denoted by $\overline{(g,x_1,\dots,x_k)}$. The sets $\mathcal{B}^{(k)}$ and $\mathcal{N}^{(k)}$ are the images of $G\times\mathfrak{b}^k$ and $G\times\mathfrak{u}^k$ respectively by the map $(g,x_1,\dots,x_k)\mapsto(g(x_1),\dots,g(x_k))$ so that $\mathcal{B}^{(k)}$ and $\mathcal{N}^{(k)}$ are closed subsets of $\mathfrak{g}^k$ by Lemma 1.7. Let $\mathcal{B}^{(k)}_{\rm n}$ be the normalization of $\mathcal{B}^{(k)}$ and let $\eta_{\rm n}$ be the normalization morphism. The map

$$G\times\mathfrak{b}^k \longrightarrow \mathcal{B}^{(k)}\ , \qquad (g,x_1,\dots,x_k)\longrightarrow(g(x_1),\dots,g(x_k))$$

defines through the quotient a morphism $\gamma :\ G\times_B\mathfrak{b}^k \longrightarrow \mathcal{B}^{(k)}$ and we have the commutative diagram:

$$\begin{tikzcd} G\times_B\mathfrak{b}^k \arrow[rr, "\gamma_{\rm n}"] \arrow[dr, "\gamma"'] & & \mathcal{B}^{(k)}_{\rm n} \arrow[dl, "\eta_{\rm n}"] \\ & \mathcal{B}^{(k)} & \end{tikzcd}$$

where $\gamma_{\rm n}$ is uniquely defined by this diagram. Let $\mathcal{N}^{(k)}_{\rm n}$ be the normalization of $\mathcal{N}^{(k)}$ and let $\varkappa$ be the normalization morphism. We have the commutative diagram:

$$\begin{tikzcd} G\times_B\mathfrak{u}^k \arrow[rr, "\upsilon_{\rm n}"] \arrow[dr, "\upsilon"'] & & \mathcal{N}^{(k)}_{\rm n} \arrow[dl, "\varkappa"] \\ & \mathcal{N}^{(k)} & \end{tikzcd}$$

with υ the restriction of γ to $G\times_B\mathfrak{u}^k$ and $\upsilon_{\rm n}$ is uniquely defined by this diagram.

• Let i be the injection $(x_1,\dots,x_k)\mapsto\overline{(1_{\mathfrak{g}},x_1,\dots,x_k)}$ from $\mathfrak{b}^k$ to $G\times_B\mathfrak{b}^k$. Then $\iota := \gamma\circ i$ is the identity of $\mathfrak{b}^k$ and $\iota_{\rm n} := \gamma_{\rm n}\circ i$ is a closed embedding of $\mathfrak{b}^k$ into $\mathcal{B}^{(k)}_{\rm n}$. In particular, $\mathcal{B}^{(k)} = G.\iota(\mathfrak{b}^k)$ and $\mathcal{B}^{(k)}_{\rm n} = G.\iota_{\rm n}(\mathfrak{b}^k)$.

• Let e be the sum of the x_β's, β in Π, and let h be the element of $\mathfrak{h}\cap[\mathfrak{g},\mathfrak{g}]$ such that $\beta(h)=2$ for all β in Π. Then there exists a unique f in $[\mathfrak{g},\mathfrak{g}]$ such that (e,h,f) is a principal $\mathfrak{sl}_2$-triple. The one-parameter subgroup of G generated by ${\rm ad}\,h$ is denoted by $t\mapsto h(t)$. The Borel subalgebra containing f is denoted by $\mathfrak{b}_-$ and its nilpotent radical is denoted by $\mathfrak{u}_-$. Let B_- be the normalizer of $\mathfrak{b}_-$ in G and let U and U_- be the unipotent radicals of B and B_- respectively.

Lemma 1.8. *Let $k\geq 2$ be an integer. Let X be an affine variety and set $Y := \mathfrak{b}^k\times X$. Let Z be a closed B-invariant subset of Y under the group action given by $g.(v_1,\dots,v_k,x) = (g(v_1),\dots,g(v_k),x)$ with $(g,v_1,\dots,v_k)$ in $B\times\mathfrak{b}^k$ and x in X. Then $Z\cap\mathfrak{h}^k\times X$ is the image of Z by the projection $(v_1,\dots,v_k,x)\mapsto(\overline{v_1},\dots,\overline{v_k},x)$.*

Proof. For all v in $\mathfrak{b}$,

$$\overline{v} = \lim_{t\to 0} h(t)(v)$$

whence the lemma since Z is closed and B-invariant. □

• For $x \in \mathfrak{g}$, let $x_{\rm s}$ and $x_{\rm n}$ be the semisimple and nilpotent components of x in $\mathfrak{g}$. Denote by $\mathfrak{g}^x$ and G^x the centralizers of x in $\mathfrak{g}$ and G respectively. For $\mathfrak{a}$ a subalgebra of $\mathfrak{g}$ and for A a subgroup of G, set:

$$\mathfrak{a}^x := \mathfrak{a} \cap \mathfrak{g}^x \qquad A^x := A \cap G^x$$

The set of regular elements of $\mathfrak{g}$ is

$$\mathfrak{g}_{\rm reg} \ := \ \{x \in \mathfrak{g} \mid \dim \mathfrak{g}^x = \ell\}$$

and denote by $\mathfrak{g}_{\rm reg,ss}$ the set of regular semisimple elements of $\mathfrak{g}$. Both $\mathfrak{g}_{\rm reg}$ and $\mathfrak{g}_{\rm reg,ss}$ are G-invariant dense open subsets of $\mathfrak{g}$. Setting $\mathfrak{h}_{\rm reg} := \mathfrak{h} \cap \mathfrak{g}_{\rm reg}$, $\mathfrak{b}_{\rm reg} := \mathfrak{b} \cap \mathfrak{g}_{\rm reg}$, $\mathfrak{u}_{\rm reg} := \mathfrak{u} \cap \mathfrak{g}_{\rm reg}$, $\mathfrak{g}_{\rm reg,ss} = G(\mathfrak{h}_{\rm reg})$, $\mathfrak{g}_{\rm reg} = G(\mathfrak{b}_{\rm reg})$ and $G(\mathfrak{u}_{\rm reg})$ is the set of regular elements of the nilpotent cone $\mathfrak{N}_{\mathfrak{g}}$ of $\mathfrak{g}$.

Lemma 1.9. *Let $k \geq 2$ be an integer and let x be in $\mathfrak{g}^k$. For O open subset of $\mathfrak{g}_{\rm reg}$, $E_x \cap O$ is not empty if and only if for some g in $\mathrm{GL}_k(\Bbbk)$, the first component of $g.x$ is in O.*

Proof. Since the components of $g.x$ are in E_x for all g in $\mathrm{GL}_k(\Bbbk)$, the condition is sufficient. Suppose that $E_x \cap O$ is not empty and denote by $x_1, \ldots, x_k$ the components of x. For some $(a_1, \ldots, a_k)$ in $\Bbbk^k \setminus \{0\}$,

$$a_1x_1 + \cdots + a_kx_k \in O$$

Let i be such that $a_i \neq 0$ and let τ be the transposition such that $\tau(1) = i$. Denoting by g the element of $\mathrm{GL}_k(\Bbbk)$ such that $g_{1,j} = a_{\tau(j)}$ for $j = 1, \ldots, k$, $g_{j,j} = 1$ for $j = 2, \ldots, k$ and $g_{j,l} = 0$ for $j \geq 2$ and $j \neq l$, the first component of $g\tau.x$ is in O. □

• Denote by $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ the algebra of $\mathfrak{g}$-invariant elements of $\mathrm{S}(\mathfrak{g})$. Let $p_1, \ldots, p_\ell$ be homogeneous generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ of degree $d_1, \ldots, d_\ell$ respectively. Choose the polynomials $p_1, \ldots, p_\ell$ so that $d_1 \leq \cdots \leq d_\ell$. For $i = 1, \ldots, \ell$ and $(x, y) \in \mathfrak{g} \times \mathfrak{g}$, consider a shift of p_i in the direction y: $p_i(x + ty)$ with $t \in \Bbbk$. Expanding $p_i(x + ty)$ as a polynomial in t, we obtain

$$p_i(x+ty) = \sum_{m=0}^{d_i} p_i^{(m)}(x,y)t^m; \qquad \forall (t,x,y) \in \Bbbk \times \mathfrak{g} \times \mathfrak{g} \tag{1}$$

where $y \mapsto (m!)p_i^{(m)}(x, y)$ is the derivative at x of p_i at the order m in the direction y. The elements $p_i^{(m)}$ defined by (1) are invariant elements of $\mathrm{S}(\mathfrak{g}) \otimes_{\Bbbk} \mathrm{S}(\mathfrak{g})$ under the diagonal action of G in $\mathfrak{g} \times \mathfrak{g}$. Remark that $p_i^{(0)}(x, y) = p_i(x)$ while $p_i^{(d_i)}(x, y) = p_i(y)$ for all $(x, y) \in \mathfrak{g} \times \mathfrak{g}$.

Remark 1.10. The family $\mathcal{P}_x := \{p_i^{(m)}(x, .);\ 1 \leq i \leq \ell, 1 \leq m \leq d_i\}$ for $x \in \mathfrak{g}$, is a Poisson-commutative family of $\mathrm{S}(\mathfrak{g})$ by Mishchenko-Fomenko [MF78]. We say that the family $\mathcal{P}_x$ is constructed by the *argument shift method*.

• Let $i \in \{1, \ldots, \ell\}$. For x in $\mathfrak{g}$, denote by $\varepsilon_i(x)$ the element of $\mathfrak{g}$ given by

$$\langle \varepsilon_i(x), y \rangle = \frac{\mathrm{d}}{\mathrm{d}t} p_i(x + ty)\,|_{t=0}$$

for all y in $\mathfrak{g}$. Thereby, ε_i is an invariant element of $\mathrm{S}(\mathfrak{g}) \otimes_{\Bbbk} \mathfrak{g}$ under the canonical action of G. According to [Ko63, Theorem 9], for x in $\mathfrak{g}$, x is in $\mathfrak{g}_{\rm reg}$ if and only if $\varepsilon_1(x), \ldots, \varepsilon_\ell(x)$ are linearly independent. In this case, $\varepsilon_1(x), \ldots, \varepsilon_\ell(x)$ is a basis of $\mathfrak{g}^x$.

Denote by $\mathfrak{z}_{\mathfrak{g}}$ the center of $\mathfrak{g}$ and for x in $\mathfrak{g}$ by $\mathfrak{z}_x$ the center of $\mathfrak{g}^x$. As $\varepsilon_1, \ldots, \varepsilon_\ell$ are invariant, for all x in $\mathfrak{g}$, $\varepsilon_1(x), \ldots, \varepsilon_\ell(x)$ are in $\mathfrak{z}_x$.

• Denote by $\varepsilon_i^{(m)}$, for $0 \leq m \leq d_i - 1$, the elements of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$ defined by the equality:

$$\varepsilon_i(x+ty) = \sum_{m=0}^{d_i-1} \varepsilon_i^{(m)}(x,y)t^m, \qquad \forall (t,x,y) \in \Bbbk\times\mathfrak{g}\times\mathfrak{g} \tag{2}$$

and set:

$$V_{x,y} := \mathrm{span}(\{\varepsilon_i^{(0)}(x,y),\ldots,\varepsilon_i^{(d_i-1)}(x,y),\ i=1,\ldots,\ell\})$$

for (x,y) in $\mathfrak{g}\times\mathfrak{g}$. According to [Bol91, Corollary 2], $V_{x,y}$ has dimension $\mathrm{b}_{\mathfrak{g}}$ if and only if $E_{x,y}$ has dimension 2 and $E_{x,y}\setminus\{0\}$ is contained in $\mathfrak{g}_{\mathrm{reg}}$.

2. On the varieties $\mathcal{B}^{(k)}$

Let $k \geq 2$ be an integer. According to the above notations, we have the commutative diagrams:

$$\begin{tikzcd} G\times_B \mathfrak{b}^k \arrow[rr, "\gamma_{\mathrm{n}}"] \arrow[dr, "\gamma"'] & & \mathcal{B}_{\mathrm{n}}^{(k)} \arrow[dl, "\eta_{\mathrm{n}}"] \\ & \mathcal{B}^{(k)} & \end{tikzcd} \qquad \begin{tikzcd} G\times_B \mathfrak{u}^k \arrow[rr, "\upsilon_{\mathrm{n}}"] \arrow[dr, "\upsilon"'] & & \mathcal{N}_{\mathrm{n}}^{(k)} \arrow[dl, "\varkappa"] \\ & \mathcal{N}^{(k)} & \end{tikzcd}$$

Since the Borel subalgebras of $\mathfrak{g}$ are conjugate under G, $\mathcal{B}^{(k)}$ is the subset of elements of $\mathfrak{g}^k$ whose components are in a same Borel subalgebra and $\mathcal{N}^{(k)}$ are the elements of $\mathcal{B}^{(k)}$ whose all the components are nilpotent.

Lemma 2.1. (i) *The morphism γ from $G\times_B\mathfrak{b}^k$ to $\mathcal{B}^{(k)}$ is projective and birational. In particular, $G\times_B\mathfrak{b}^k$ is a desingularization of $\mathcal{B}^{(k)}$ and $\mathcal{B}^{(k)}$ has dimension $k\mathrm{b}_{\mathfrak{g}}+n$.*

(ii) *The morphism υ from $G\times_B\mathfrak{u}^k$ to $\mathcal{N}^{(k)}$ is projective and birational. In particular, $G\times_B\mathfrak{u}^k$ is a desingularization of $\mathcal{N}^{(k)}$ and $\mathcal{N}^{(k)}$ has dimension $(k+1)n$.*

Proof. (i) Denote by $\Omega_{\mathfrak{g}}^{(2)}$ the subset of elements (x,y) of $\mathfrak{g}^2$ such that $E_{x,y}$ has dimension 2 and such that $E_{x,y}\setminus\{0\}$ is contained in $\mathfrak{g}_{\mathrm{reg}}$. According to Lemma 1.7, γ is a projective morphism. For $1\leq i<j\leq k$, let $\Omega_{i,j}^{(k)}$ be the inverse image of $\Omega_{\mathfrak{g}}^{(2)}$ by the projection

$$(x_1,\ldots,x_k) \longmapsto (x_i,x_j)$$

Then $\Omega_{i,j}^{(k)}$ is an open subset of $\mathfrak{g}^k$ whose intersection with $\mathcal{B}^{(k)}$ is not empty. Let $\Omega_{\mathfrak{g}}^{(k)}$ be the union of the $\Omega_{i,j}^{(k)}$. According to [Bol91, Corollary 2] and [Ko63, Theorem 9], for (x,y) in $\Omega_{\mathfrak{g}}^{(2)}\cap\mathcal{B}^{(2)}$, $V_{x,y}$ is the unique Borel subalgebra of $\mathfrak{g}$ containing x and y so that the restriction of γ to $\gamma^{-1}(\Omega_{\mathfrak{g}}^{(k)})$ is a bijection onto $\Omega_{\mathfrak{g}}^{(k)}$. Hence γ is birational. Moreover, $G\times_B\mathfrak{b}^k$ is a smooth variety as a vector bundle over the smooth variety G/B, whence the assertion since $G\times_B\mathfrak{b}^k$ has dimension $k\mathrm{b}_{\mathfrak{g}}+n$.

(ii) According to Lemma 1.7, υ is a projective morphism. Let $\mathcal{N}_{\mathrm{reg}}^{(k)}$ be the subset of elements of $\mathcal{N}^{(k)}$ whose at least one component is a regular element of $\mathfrak{g}$. Then $\mathcal{N}_{\mathrm{reg}}^{(k)}$ is an open subset of $\mathcal{N}^{(k)}$. Since a regular nilpotent element is contained in one and only one Borel subalgebra of $\mathfrak{g}$, the restriction of υ to $\upsilon^{-1}(\mathcal{N}_{\mathrm{reg}}^{(k)})$ is a bijection onto $\mathcal{N}_{\mathrm{reg}}^{(k)}$. Hence υ is birational. Moreover, $G\times_B\mathfrak{u}^k$ is a smooth variety as a vector bundle over the smooth variety G/B, whence the assertion since $G\times_B\mathfrak{u}^k$ has dimension $(k+1)n$. ∎

2.1. Let κ be the map

$$U_{-} \times \mathfrak{u}_{\rm reg} \longrightarrow \mathfrak{N}_{\mathfrak{g}} \qquad (g,x) \longmapsto g(x)$$

Lemma 2.2. *Let V be the set of elements of $\mathcal{N}^{(k)}$ whose first component is in $U_{-}(\mathfrak{u}_{\rm reg})$ and let V_k be the set of elements x of $\mathcal{N}^{(k)}$ such that $E_x \cap \mathfrak{g}_{\rm reg}$ is not empty.*

(i) *The image of κ is a smooth open subset of $\mathfrak{N}_{\mathfrak{g}}$ and κ is an ismorphism onto $U_{-}(\mathfrak{u}_{\rm reg})$.*

(ii) *The subset V of $\mathcal{N}^{(k)}$ is open.*

(iii) *The open subset V of $\mathcal{N}^{(k)}$ is smooth.*

(iv) *The set V_k is a smooth open subset of $\mathcal{N}^{(k)}$.*

Proof. (i) Since $\mathfrak{N}_{\mathfrak{g}}$ is the nullvariety of $p_1,\ldots,p_\ell$ in $\mathfrak{g}$, $\mathfrak{N}_{\mathfrak{g}} \cap \mathfrak{g}_{\rm reg}$ is a smooth open susbet of $\mathfrak{N}_{\mathfrak{g}}$ by [Ko63, Theorem 9]. For (g,x) in $U_{-} \times \mathfrak{u}_{\rm reg}$ such that $g(x)$ is in $\mathfrak{u}$, $b^{-1}g$ is in G^x for some b in B since $B(x) = \mathfrak{u}_{\rm reg}$. Hence $g = 1_{\mathfrak{g}}$ since G^x is contained in B and since $U_{-} \cap B = \{1_{\mathfrak{g}}\}$. As a result, κ is an injective morphism from the smooth variety $U_{-} \times \mathfrak{u}_{\rm reg}$ to the smooth variety $\mathfrak{N}_{\mathfrak{g}} \cap \mathfrak{g}_{\rm reg}$. Hence κ is an open immersion by Zariski's Main Theorem [Mu88, §9].

(ii) By definition, V is the intersection of $\mathcal{N}^{(k)}$ and $U_{-}(\mathfrak{u}_{\rm reg}) \times \mathfrak{N}_{\mathfrak{g}}^{k-1}$. So, by (i), it is an open subset of $\mathcal{N}^{(k)}$.

(iii) Let $(x_1,\ldots,x_k)$ be in $\mathfrak{u}^k$ and let g be in G such that $(g(x_1),\ldots,g(x_k))$ is in V. Then x_1 is in $\mathfrak{u}_{\rm reg}$ and for some (g',b) in $U_{-} \times B$, $g'b(x_1) = g(x_1)$. Hence $g^{-1}g'b$ is in G^{x_1} and g is in $U_{-}B$ since G^{x_1} is contained in B. As a result, the map

$$U_{-} \times \mathfrak{u}_{\rm reg} \times \mathfrak{u}^{k-1} \longrightarrow V \qquad (g,x_1,\ldots,x_k) \longmapsto (g(x_1),\ldots,g(x_k))$$

is an isomorphism whose inverse is given by

$$V \longrightarrow U_{-} \times \mathfrak{u}_{\rm reg} \times \mathfrak{u}^{k-1} \qquad (x_1,\ldots,x_k) \longmapsto (\kappa^{-1}(x_1)_1, \kappa^{-1}(x_1)_1(x_1),\ldots,\kappa^{-1}(x_1)_1(x_k))$$

with κ^{-1} the inverse of κ and $\kappa^{-1}(x_1)_1$ the component of $\kappa^{-1}(x_1)$ on U_{-}, whence the assertion since $U_{-} \times \mathfrak{u}_{\rm reg} \times \mathfrak{u}^{k-1}$ is smooth.

(iv) According to Lemma 1.9, $V_k = {\rm GL}_k(\Bbbk).V$, whence the assertion by (iii). □

Corollary 2.3. (i) *The subvariety $\mathcal{N}^{(k)} \setminus V_k$ of $\mathcal{N}^{(k)}$ has codimension $k+1$.*

(ii) *The restriction of υ to $\upsilon^{-1}(V_k)$ is an isomorphism onto V_k.*

(iii) *The subset $\upsilon^{-1}(V_k)$ is a big open subset of $G \times_B \mathfrak{u}^k$.*

Proof. (i) By definition, $\mathcal{N}^{(k)} \setminus V_k$ is the subset of elements x of $\mathcal{N}^{(k)}$ such that E_x is contained in $\mathfrak{g} \setminus \mathfrak{g}_{\rm reg}$. Hence $\mathcal{N}^{(k)} \setminus V_k$ is contained in the image of $G \times_B (\mathfrak{u} \setminus \mathfrak{u}_{\rm reg})^k$ by υ. Let $(x_1,\ldots,x_k)$ be in $\mathfrak{u}^k \cap (\mathcal{N}^{(k)} \setminus V_k)$. Then, for all $(a_1,\ldots,a_k)$ in $\Bbbk^k$,

$$\langle x_{-\beta}, a_1x_1 + \cdots + a_kx_k \rangle = 0$$

for some β in Π. Since Π is finite, E_x is orthogonal to $x_{-\beta}$ for some β in Π. As a result, the subvariety of Borel subalgebras of $\mathfrak{g}$ containing $x_1,\ldots,x_k$ has positive dimension. Hence

$$\dim(\mathcal{N}^{(k)} \setminus V_k) < \dim G \times_B (\mathfrak{u} \setminus \mathfrak{u}_{\rm reg})^k = n + k(n-1)$$

Moreover, for β in Π, denoting by $\mathfrak{u}_\beta$ the orthogonal complement of $\mathfrak{g}^{-\beta}$ in $\mathfrak{u}$, $\upsilon(G \times_B (\mathfrak{u}_\beta)^k)$ is contained in $\mathcal{N}^{(k)} \setminus V_k$ and its dimension equal $(k+1)(n-1)$ since the variety of Borel subalgebras containing $\mathfrak{u}_\beta$ has dimension 1, whence the assertion.

(ii) For x in $\mathcal{N}^{(k)}$, E_x is contained in all Borel subalgebra of $\mathfrak{g}$, containing the components of x. Then the restriction of υ to $\upsilon^{-1}(V_k)$ is injective since all regular nilpotent element of $\mathfrak{g}$ is contained in a single Borel subalgebra of $\mathfrak{g}$, whence the assertion by Zariski's Main Theorem [Mu88, §9] since V_k is a smooth open subset of $\mathcal{N}^{(k)}$ by Lemma 2.2,(iv).

(iii) Identify U_- with the open subset U_-B/B of G/B and denote by π_0 the bundle projection from $G\times_B \mathfrak{u}^k \longrightarrow G/B$. Since $\upsilon^{-1}(V_k)$ is G-invariant, it suffices to prove that $\upsilon^{-1}(V_k)\cap\pi_0^{-1}(U_-)$ is a big open subset of $\pi_0^{-1}(U_-)$. Let V_0 be the subset of elements x of $\mathfrak{u}^k$ such that $E_x\cap\mathfrak{g}_{\rm reg}$ is not empty. Then $\mathfrak{u}^k\setminus V_0$ is contained in $(\mathfrak{u}\setminus\mathfrak{u}_{\rm reg})^k$ and has codimension at least 2 in $\mathfrak{u}^k$ since $k\geq 2$. As a result, $U_-\times V_0$ is a big open subset of $U_-\times\mathfrak{u}^k$. The open subset $\pi_0^{-1}(U_-)$ of $G\times_B\mathfrak{u}^k$ identifies with $U_-\times\mathfrak{u}^k$ and $\upsilon^{-1}(V_k)\cap\pi_0^{-1}(U_-)$ identifies with $U_-\times V_0$, whence the assertion. □

2.2. Denote by $\pi_{\mathfrak{g}}\ :\ \mathfrak{g}\longrightarrow\mathfrak{g}//G$ and $\pi_{\mathfrak{h}}\ :\ \mathfrak{h}\longrightarrow\mathfrak{h}/W(\mathcal{R})$ the quotient maps, i.e. the morphisms defined by the invariants. Recall $\mathfrak{g}//G=\mathfrak{h}/W(\mathcal{R})$, and let $\mathcal{X}$ be the following fiber product:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\overline{\chi}"] \arrow[d, "\overline{\rho}"'] & \mathfrak{g} \arrow[d, "\pi_{\mathfrak{g}}"] \\ \mathfrak{h} \arrow[r, "\pi_{\mathfrak{h}}"'] & \mathfrak{h}/W(\mathcal{R}) \end{tikzcd}$$

where $\overline{\chi}$ and $\overline{\rho}$ are the restriction maps. The actions of G and $W(\mathcal{R})$ on $\mathfrak{g}$ and $\mathfrak{h}$ respectively induce an action of $G\times W(\mathcal{R})$ on $\mathcal{X}$: $(g,w).(x,y):=(g(x),w(y))$.

Lemma 2.4. (i) *There exists a well defined G-equivariant morphism $\chi_{\rm n}$ from $G\times_B\mathfrak{b}$ to $\mathcal{X}$ such that γ is the composition of $\chi_{\rm n}$ and $\overline{\chi}$.*

(ii) *The morphism $\chi_{\rm n}$ is projective and birational. Moreover, $\mathcal{X}$ is irreducible.*

(iii) *The subscheme $\mathcal{X}$ is normal. Moreover, every element of $\mathfrak{g}_{\rm reg}\times\mathfrak{h}\cap\mathcal{X}$ is a smooth point of $\mathcal{X}$.*

(iv) *The algebra $\Bbbk[\mathcal{X}]$ is the space of global sections of $\mathcal{O}_{G\times_B\mathfrak{b}}$ and $\Bbbk[\mathcal{X}]^G=\mathrm{S}(\mathfrak{h})$.*

Proof. (i) Since the map $(g,x)\mapsto(g(x),\overline{x})$ is constant on the B-orbits, there exists a uniquely defined morphism $\chi_{\rm n}$ from $G\times_B\mathfrak{b}$ to $\mathfrak{g}\times\mathfrak{h}$ such that $(g(x),\overline{x})$ is the image by $\chi_{\rm n}$ of the image of (g,x) in $G\times_B\mathfrak{b}$. The image of $\chi_{\rm n}$ is contained in $\mathcal{X}$ since for all p in $\mathrm{S}(\mathfrak{g})^G$, $p(\overline{x})=p(x)=p(g(x))$. Furthermore, $\chi_{\rm n}$ verifies the condition of the assertion.

(ii) According to Lemma 1.7, $\chi_{\rm n}$ is a projective morphism. Let (x,y) be in $\mathfrak{g}\times\mathfrak{h}$ such that $p(x)=p(y)$ for all p in $\mathrm{S}(\mathfrak{g})^G$. For some g in G, $g(x)$ is in $\mathfrak{b}$ and its semisimple component is y so that (x,y) is in the image of $\chi_{\rm n}$. As a result, $\mathcal{X}$ is irreducible as the image of the irreducible variety $G\times_B\mathfrak{b}$. Since for all (x,y) in $\mathcal{X}\cap\mathfrak{h}_{\rm reg}\times\mathfrak{h}_{\rm reg}$, there exists a unique w in $W(\mathcal{R})$ such that $y=w(x)$, the fiber of $\chi_{\rm n}$ at any element $\mathcal{X}\cap G.(\mathfrak{h}_{\rm reg}\times\mathfrak{h}_{\rm reg})$ has one element. Hence $\chi_{\rm n}$ is birational, whence the assertion.

(iii) The morphism $\pi_{\mathfrak{h}}$ is finite, and so is $\overline{\chi}$. Moreover $\pi_{\mathfrak{h}}$ is smooth over $\mathfrak{h}_{\rm reg}$, $\overline{\chi}$ is smooth over $\mathfrak{g}_{\rm reg}$. Finally, $\pi_{\mathfrak{g}}$ is flat and all fibers are normal and Cohen-Macaulay. Thus the same holds for the morphism $\overline{\rho}$. Since $\mathfrak{h}$ is smooth this implies that $\mathcal{X}$ is normal and Cohen-Macaulay by [MA86, Ch. 8, §23].

(iv) According to (ii), (iii) and Lemma 1.4, $\Bbbk[\mathcal{X}]=\mathrm{H}^0(G\times_B\mathfrak{b},\mathcal{O}_{G\times_B\mathfrak{b}})$. Under the action of G in $\mathfrak{g}\times\mathfrak{h}$, $\Bbbk[\mathfrak{g}\times\mathfrak{h}]^G=\mathrm{S}(\mathfrak{g})^G\otimes_{\Bbbk}\mathrm{S}(\mathfrak{h})$ and its image in $\Bbbk[\mathcal{X}]$ by the quotient morphism is equal to $\mathrm{S}(\mathfrak{h})$. Moreover, since G is reductive, $\Bbbk[\mathcal{X}]^G$ is the image of $\Bbbk[\mathfrak{g}\times\mathfrak{h}]^G$ by the quotient morphism, whence the assertion. □

Proposition 2.5. [He76, Theorem B and Corollary] *The variety $\mathcal{X}$ has rational singularities.*

Corollary 2.6. (i) *Let x and x' be in $\mathfrak{b}_{\rm reg}$ such that x' is in $G.x$ and $\overline{x'}=\overline{x}$. Then x' is in $B(x)$.*

(ii) *For all w in $W(\mathcal{R})$, the map*

$$U_-\times\mathfrak{b}_{\rm reg}\longrightarrow\mathcal{X},\qquad (g,x)\longmapsto(g(x),w(\overline{x}))$$

is an isomorphism onto a smooth open subset of $\mathcal{X}$*.*

Proof. (i) The semisimple components of x and x' are conjugate under B since they are conjugate to $\overline{x}$ under B. Let b and b' be in B such that $\overline{x}$ is the semisimple component of $b(x)$ and $b'(x')$. Then the nilpotent components of $b(x)$ and $b'(x')$ are regular nilpotent elements of $\mathfrak{g}^{\overline{x}}$, belonging to the Borel subalgebra $\mathfrak{b} \cap \mathfrak{g}^{\overline{x}}$ of $\mathfrak{g}^{\overline{x}}$. Hence x' is in $B(x)$ since regular nilpotent elements of a Borel subalgebra of a reductive Lie algebra are conjugate under the correponding Borel subgroup.

(ii) Since the action of G and $W(\mathcal{R})$ on $\mathcal{X}$ commute, it suffices to prove the assertion for $w = 1_{\mathfrak{h}}$. Denote by θ the map

$$U_{-} \times \mathfrak{b}_{\rm reg} \longrightarrow \mathcal{X} , \qquad (g, x) \longmapsto (g(x), \overline{x}).$$

Let (g, x) and (g', x') be in $U_{-} \times \mathfrak{b}_{\rm reg}$ such that $\theta(g, x) = \theta(g', x')$. By (i), $x' = b(x)$ for some b in B. Hence $g^{-1}g'b$ is in G^x. Since x is in $\mathfrak{b}_{\rm reg}$, G^x is contained in B and $g^{-1}g'$ is in $U_{-} \cap B$, whence $(g, x) = (g', x')$ since $U_{-} \cap B = \{1_{\mathfrak{g}}\}$. As a result, θ is a dominant injective map from $U_{-} \times \mathfrak{b}_{\rm reg}$ to the normal variety $\mathcal{X}$. Hence θ is an isomorphism onto a smooth open subset of $\mathcal{X}$, by Zariski's Main Theorem [Mu88, §9]. $\square$

2.3. According to Lemma 2.1,(i), $G \times_B \mathfrak{b}^k$ is a desingularization of $\mathcal{B}^{(k)}$ and we have the commutative diagram:

$$\begin{tikzcd} G \times_B \mathfrak{b}^k \arrow[rr, "\gamma_{\rm n}"] \arrow[dr, "\gamma"'] & & \mathcal{B}_{\rm n}^{(k)} \arrow[dl, "\eta_{\rm n}"] \\ & \mathcal{B}^{(k)} & \end{tikzcd}$$

Lemma 2.7. *Let* ϖ *be the canonical projection from* $\mathcal{X}^k$ *to* $\mathfrak{g}^k$*. Denote by* ι_k *the map*

$$\mathfrak{b}^k \longrightarrow \mathcal{X}^k, \qquad (x_1, \dots, x_k) \longmapsto (x_1, \dots, x_k, \overline{x_1}, \dots, \overline{x_k}).$$

(i) *The map* ι_k *is a closed embedding of* $\mathfrak{b}^k$ *into* $\mathcal{X}^k$*.*

(ii) *The subvariety* $\iota_k(\mathfrak{b}^k)$ *of* $\mathcal{X}^k$ *is an irreducible component of* $\varpi^{-1}(\mathfrak{b}^k)$*.*

(iii) *The subvariety* $\varpi^{-1}(\mathfrak{b}^k)$ *of* $\mathcal{X}^k$ *is invariant under the canonical action of* $W(\mathcal{R})^k$ *in* $\mathcal{X}^k$ *and this action induces a simply transitive action of* $W(\mathcal{R})^k$ *on the set of irreducible components of* $\varpi^{-1}(\mathfrak{b}^k)$*.*

Proof. (i) The map

$$\mathfrak{b}^k \longrightarrow G^k \times \mathfrak{b}^k , \qquad (x_1, \dots, x_k) \longmapsto (1_{\mathfrak{g}}, \dots, 1_{\mathfrak{g}}, x_1, \dots, x_k)$$

defines through the quotient a closed embedding of $\mathfrak{b}^k$ in $G^k \times_{B^k} \mathfrak{b}^k$. Denote it by ι'. Let $\chi_{\rm n}^{(k)}$ be the map

$$G^k \times_{B^k} \mathfrak{b}^k \longrightarrow \mathcal{X}^k , \qquad (x_1, \dots, x_k) \longmapsto (\chi_{\rm n}(x_1), \dots, \chi_{\rm n}(x_k)).$$

Then $\iota_k = \chi_{\rm n}^{(k)} \circ \iota'$. Since $\chi_{\rm n}$ is a projective morphism, ι_k is a closed morphism. Moreover, it is an embedding since $\varpi \circ \iota_k$ is the identity of $\mathfrak{b}^k$.

(ii) Since $\mathrm{S}(\mathfrak{h})$ is a finite extension of $\mathrm{S}(\mathfrak{h})^{W(\mathcal{R})}$, ϖ is a finite morphism. So $\varpi^{-1}(\mathfrak{b}^k)$ and $\mathfrak{b}^k$ have the same dimension. According to (i), $\iota_k(\mathfrak{b}^k)$ is an irreducible subvariety of $\omega^{-1}(\mathfrak{b}^k)$ of the same dimension, whence the assertion.

(iii) Since all the fibers of ϖ are invariant under the action of $W(\mathcal{R})^k$ on $\mathcal{X}^k$, $\varpi^{-1}(\mathfrak{b}^k)$ is invariant under this action and $W(\mathcal{R})^k$ permutes the irreducible components of $\varpi^{-1}(\mathfrak{b}^k)$. For w in $W(\mathcal{R})^k$, set

$Z_w := w.\iota_k(\mathfrak{b}^k)$. If $Z_w = \iota_k(\mathfrak{b}^k)$, then for all $(x_1,\dots,x_k)$ in $\mathfrak{h}^k_{\rm reg}$, $(x_1,\dots,x_k,w.(x_1,\dots,x_k))$ is in $\iota_k(\mathfrak{b}^k)$ so that $(x_1,\dots,x_k)$ is invariant under w and w is the identity.

Let Z be an irreducible component of $\varpi^{-1}(\mathfrak{b}^k)$ and let Z_0 be its image by the map

$$(x_1,\dots,x_k,y_1,\dots,y_k) \longmapsto (\overline{x_1},\dots,\overline{x_k},y_1,\dots,y_k).$$

Since ϖ is G^k-equivariant and $\mathfrak{b}^k$ is invariant under B^k, $\varpi^{-1}(\mathfrak{b}^k)$ and Z are invariant under B^k. Hence by Lemma 1.8, Z_0 is closed. Moreover, since the image of the map

$$Z_0\times\mathfrak{u}^k \longrightarrow \mathcal{X}^k\ , \quad ((x_1,\dots,x_k,y_1,\dots,y_k),(u_1,\dots,u_k)) \longmapsto (x_1+u_1,\dots,x_k+u_k,y_1,\dots,y_k)$$

is an irreducible subset of $\varpi^{-1}(\mathfrak{b}^k)$ containing Z, Z is the image of this map. Since Z_0 is contained in $\mathcal{X}^k$, Z_0 is contained in the image of the map

$$\mathfrak{h}^k\times W(\mathcal{R})^k \longrightarrow \mathfrak{h}^k\times\mathfrak{h}^k\ , \quad (x_1,\dots,x_k,w_1,\dots,w_k) \longmapsto (x_1,\dots,x_k,w_1(x_1),\dots,w_k(x_k)).$$

As $W(\mathcal{R})$ is finite and Z_0 is irreducible, for some w in $W(\mathcal{R})^k$, Z_0 is the image of $\mathfrak{h}^k$ by the map

$$(x_1,\dots,x_k) \longmapsto (x_1,\dots,x_k,w.(x_1,\dots,x_k))$$

and $Z = Z_w$, whence the assertion. □

Set $\mathfrak{Y} := G^k\times_{B^k}\mathfrak{b}^k$. The map

$$G\times\mathfrak{b}^k \longrightarrow G^k\times\mathfrak{b}^k\ , \qquad (g,v_1,\dots,v_k) \longmapsto (g,\dots,g,v_1,\dots,v_k)$$

defines through the quotient a closed immersion from $G\times_B\mathfrak{b}^k$ to $\mathfrak{Y}$. Denote it by ν. Consider the diagonal action of G on $\mathfrak{g}^k$ and $\mathcal{X}^k$: $g.(x_1,\dots,x_k,y_1,\dots,y_k) := (g(x_1),\dots,g(x_k),y_1,\dots,y_k)$, and identify $G\times_B\mathfrak{b}^k$ with $\nu(G\times_B\mathfrak{b}^k)$ by the closed immersion ν.

Corollary 2.8. *Set* $\mathcal{B}^{(k)}_{\rm x} := G.\iota_k(\mathfrak{b}^k)$.

(i) *The subset* $\mathcal{B}^{(k)}_{\rm x}$ *is the image of* $G\times_B\mathfrak{b}^k$ *by* $\chi^{(k)}_{\rm n}$. *Moreover, the restriction of* $\chi^{(k)}_{\rm n}$ *to* $G\times_B\mathfrak{b}^k$ *is a projective birational morphism from* $G\times_B\mathfrak{b}^k$ *onto* $\mathcal{B}^{(k)}_{\rm x}$.

(ii) *The subset* $\mathcal{B}^{(k)}_{\rm x}$ *of* $\mathcal{X}^k$ *is an irreducible component of* $\varpi^{-1}(\mathcal{B}^{(k)})$.

(iii) *The subvariety* $\varpi^{-1}(\mathcal{B}^{(k)})$ *of* $\mathcal{X}^k$ *is invariant under* $W(\mathcal{R})^k$ *and this action induces a simply transitive action of* $W(\mathcal{R})^k$ *on the set of irreducible components of* $\varpi^{-1}(\mathcal{B}^{(k)})$.

(iv) *The subalgebra* $\Bbbk[\mathcal{B}^{(k)}]$ *of* $\Bbbk[\varpi^{-1}(\mathcal{B}^{(k)})]$ *equals* $\Bbbk[\varpi^{-1}(\mathcal{B}^{(k)})]^{W(\mathcal{R})^k}$ *with respect to the action of* $W(\mathcal{R})^k$ *on* $\varpi^{-1}(\mathcal{B}^{(k)})$.

Proof. (i) Let $\gamma_{\rm x}$ be the restriction of $\chi^{(k)}_{\rm n}$ to $G\times_B\mathfrak{b}^k$. Since $\iota_k = \chi^{(k)}_{\rm n}\circ\iota'$, $G\times_B\mathfrak{b}^k = G.\iota'(\mathfrak{b}^{(k)})$ and $\chi^{(k)}_{\rm n}$ is G-equivariant, $\mathcal{B}^{(k)}_{\rm x} = \gamma_{\rm x}(G\times_B\mathfrak{b}^k)$. Hence $\mathcal{B}^{(k)}_{\rm x}$ is closed in $\mathcal{X}^k$ and $\gamma_{\rm x}$ is a projective morphism from $G\times_B\mathfrak{b}^k$ to $\mathcal{B}^{(k)}_{\rm x}$ since $\chi^{(k)}_{\rm n}$ is a projective morphism. According to Lemma 2.1,(i), $\varpi\circ\gamma_{\rm x}$ is a birational morphism onto $\mathcal{B}^{(k)}$. Then $\gamma_{\rm x}$ is birational since $\varpi(\mathcal{B}^{(k)}_{\rm x}) = \mathcal{B}^{(k)}$, whence the assertion.

(ii) Since ϖ is a finite morphism, $\varpi^{-1}(\mathcal{B}^{(k)})$, $\mathcal{B}^{(k)}$ and $\mathcal{B}^{(k)}_{\rm x}$ have the same dimension, whence the assertion since $\mathcal{B}^{(k)}_{\rm x}$ is irreducible as an image of an irreducible variety.

(iii) Since the fibers of ϖ are invariant under $W(\mathcal{R})^k$, $\varpi^{-1}(\mathcal{B}^{(k)})$ is invariant under this action and $W(\mathcal{R})^k$ permutes the irreducible components of $\varpi^{-1}(\mathcal{B}^{(k)})$. Let Z be an irreducible component of $\varpi^{-1}(\mathcal{B}^{(k)})$. As ϖ is G^k-equivariant, $\varpi^{-1}(\mathcal{B}^{(k)})$ and Z are invariant under the diagonal action of G. Moreover, $Z = G.(Z\cap\varpi^{-1}(\mathfrak{b}^k))$ since $\mathcal{B}^{(k)} = G.\mathfrak{b}^k$. Hence for some irreducible component Z_0 of $Z\cap\varpi^{-1}(\mathfrak{b}^k)$, $Z = G.Z_0$. According to Lemma 2.7,(iii), Z_0 is contained in $w.\iota_k(\mathfrak{b}^k)$ for some w in $W(\mathcal{R})^k$. Hence $Z = w.\mathcal{B}^{(k)}_{\rm x}$ since the actions of G^k and $W(\mathcal{R})^k$ on $\mathcal{X}^k$ commute and Z is an irreducible component of $\varpi^{-1}(\mathcal{B}^{(k)})$.

Let $w = (w_1, \ldots, w_k)$ be in $W(\mathcal{R})^k$ such that $w.\mathcal{B}_{\mathrm{x}}^{(k)} = \mathcal{B}_{\mathrm{x}}^{(k)}$. Let x be in $\mathfrak{h}_{\mathrm{reg}}$ and let $i = 1, \ldots, k$. Set:

$$z := (x_1, \ldots, x_k, \overline{x_1}, \ldots, \overline{x_k}) \text{ with } x_j := \begin{cases} x & \text{if} \quad j = i \\ x_j = e & \text{otherwise} \end{cases}.$$

Then there exists $(y_1, \ldots, y_k)$ in $\mathfrak{b}^k$ and g in G such that

$$w.z = (g(y_1), \ldots, g(y_k), \overline{y_1}, \ldots, \overline{y_k}).$$

For some b in B, $b(y_i) = \overline{y_i}$ since y_i is a regular semisimple element, belonging to $\mathfrak{b}$. As a result, $gb^{-1}(\overline{y_i}) = x$ and $w_i(x) = \overline{y_i}$. Hence gb^{-1} is an element of $N_G(\mathfrak{h})$ representing w_i^{-1}. Furthermore, since $gb^{-1}(b(y_j)) = e$ for $j \neq i$, $b(y_j)$ is a regular nilpotent element belonging to $\mathfrak{b}$. Then, since there is one and only one Borel subalgebra containing a regular nilpotent element, $gb^{-1}(\mathfrak{b}) = \mathfrak{b}$ and $w_i = 1_{\mathfrak{h}}$. As a result, w is the identity of $W(\mathcal{R})^k$, whence the assertion.

(iv) Since the fibers of ϖ are invariant under $W(\mathcal{R})^k$, $\Bbbk[\mathcal{B}^{(k)}]$ is contained in $\Bbbk[\varpi^{-1}(\mathcal{B}^{(k)})]^{W(\mathcal{R})^k}$. Let p be in $\Bbbk[\varpi^{-1}(\mathcal{B}^{(k)})]^{W(\mathcal{R})^k}$. Since $W(\mathcal{R})$ is a finite group, p is the restriction to $\varpi^{-1}(\mathcal{B}^{(k)})$ of an element q of $\Bbbk[\mathcal{X}]^{\otimes k}$, invariant under $W(\mathcal{R})^k$. Since $\Bbbk[\mathcal{X}]^{W(\mathcal{R})} = \mathrm{S}(\mathfrak{g})$, q is in $\mathrm{S}(\mathfrak{g})^{\otimes k}$ by Lemma 2.1,(iv), and p is in $\Bbbk[\mathcal{B}^{(k)}]$, whence the assertion. □

2.4. For α a positive root, denote by $\mathfrak{h}_\alpha$ the kernel of α and by S_α the closure in $\mathfrak{b}$ of the image of the map

$$U \times \mathfrak{h}_\alpha \longrightarrow \mathfrak{b}, \qquad (g, x) \longmapsto g(x).$$

For β in Π, set:

$$\mathfrak{u}_\beta := \bigoplus_{\alpha \in \mathcal{R}_+ \setminus \{\beta\}} \mathfrak{g}^\beta, \qquad \mathfrak{b}_\beta := \mathfrak{h}_\beta \oplus \mathfrak{u}_\beta.$$

Lemma 2.9. *For α in $\mathcal{R}_+$, let $\mathfrak{h}'_\alpha$ be the set of subregular elements belonging to $\mathfrak{h}_\alpha$.*

(i) *For α in $\mathcal{R}_+$, S_α is a subvariety of codimension 2 of $\mathfrak{b}$. Moreover, it is contained in $\mathfrak{b} \setminus \mathfrak{b}_{\mathrm{reg}}$.*

(ii) *For β in Π, $S_\beta = \mathfrak{b}_\beta$.*

(iii) *The S_α's, $\alpha \in \mathcal{R}_+$, are the irreducible components of $\mathfrak{b} \setminus \mathfrak{b}_{\mathrm{reg}}$.*

Proof. (i) For x in $\mathfrak{h}'_\alpha$, $\mathfrak{b}^x = \mathfrak{h} + \Bbbk x_\alpha$. Hence $U(\mathfrak{h}'_\alpha)$ has dimension $n - 1 + \ell - 1$, whence the assertion since $U(\mathfrak{h}'_\alpha)$ is dense in S_α and $\mathfrak{h}'_\alpha$ is contained in $\mathfrak{b} \setminus \mathfrak{b}_{\mathrm{reg}}$.

(ii) For β in Π, $U(\mathfrak{h}'_\beta)$ is contained in $\mathfrak{b}_\beta$ since $\mathfrak{b}_\beta$ is an ideal of $\mathfrak{b}$, whence the assertion by (i).

(iii) According to (i), it suffices to prove that $\mathfrak{b} \setminus \mathfrak{b}_{\mathrm{reg}}$ is the union of the S_α's. Let x be in $\mathfrak{b} \setminus \mathfrak{b}_{\mathrm{reg}}$. According to [V72], for some g in G and for some β in Π, x is in $g(\mathfrak{b}_\beta)$. Since $\mathfrak{b}_\beta$ is an ideal of $\mathfrak{b}$, by Bruhat's decomposition of G, for some b in B and for some w in $W(\mathcal{R})$, $b^{-1}(x)$ is in $w(\mathfrak{b}_\beta) \cap \mathfrak{b}$. By definition,

$$w(\mathfrak{b}_\beta) = w(\mathfrak{h}_\beta) \oplus w(\mathfrak{u}_\beta) = \mathfrak{h}_{w(\beta)} \oplus \bigoplus_{\alpha \in \mathcal{R}_+ \setminus \{\beta\}} \mathfrak{g}^{w(\alpha)}.$$

So,

$$w(\mathfrak{b}_\beta) \cap \mathfrak{b} = \mathfrak{h}_{w(\beta)} \oplus \mathfrak{u}_0 \text{ with } \mathfrak{u}_0 := \bigoplus_{\substack{\alpha \in \mathcal{R}_+ \setminus \{\beta\} \\ w(\alpha) \in \mathcal{R}_+}} \mathfrak{g}^{w(\alpha)}.$$

The subspace $\mathfrak{u}_0$ of $\mathfrak{u}$ is a subalgebra, not containing $\mathfrak{g}^{w(\beta)}$. Then, denoting by U_0 the closed subgroup of U whose Lie algebra is $\mathrm{ad}\, \mathfrak{u}_0$,

$$\overline{U_0(\mathfrak{h}_{w(\beta)})} = w(\mathfrak{b}_\beta) \cap \mathfrak{b}$$

since the left hand side is contained in the right hand side and has the same dimension. As a result, x is in $S_{w(\beta)}$ since $S_{w(\beta)}$ is B-invariant, whence the assertion. □

Recall that θ is the map

$$U_{-} \times \mathfrak{b}_{\mathrm{reg}} \longrightarrow \mathcal{X}, \qquad (g,x) \longmapsto (g(x), \overline{x})$$

and denote by W'_k the inverse image of $\theta(U_{-} \times \mathfrak{b}_{\mathrm{reg}})$ by the projection

$$\mathcal{B}_{\mathrm{x}}^{(k)} \longrightarrow \mathcal{X}, \qquad (x_1, \ldots, x_k, y_1, \ldots, y_k) \longmapsto (x_1, y_1).$$

Lemma 2.10. *Let W_k be the subset of elements (x,y) of $\mathcal{B}_{\mathrm{x}}^{(k)}$ ($x \in \mathfrak{g}^k, y \in \mathfrak{h}^k$) such that $E_x \cap \mathfrak{g}_{\mathrm{reg}}$ is not empty.*

(i) *The subset W'_k of $\mathcal{B}_{\mathrm{x}}^{(k)}$ is a smooth open subset. Moreover, the map*

$$U_{-} \times \mathfrak{b}_{\mathrm{reg}} \times \mathfrak{b}^{k-1} \longrightarrow W'_k, \qquad (g, x_1, \ldots, x_k) \longmapsto (g(x_1), \ldots, g(x_k), \overline{x_1}, \ldots, \overline{x_k}).$$

is an isomorphism of varieties.

(ii) *The subset $\mathcal{B}_{\mathrm{x}}^{(k)}$ of $\mathfrak{g}^k \times \mathfrak{h}^k$ is invariant under the canonical action of $\mathrm{GL}_k(\Bbbk)$.*

(iii) *The subset W_k of $\mathcal{B}_{\mathrm{x}}^{(k)}$ is a smooth open subset. Moreover, W_k is the $G \times \mathrm{GL}_k(\Bbbk)$-invariant set generated by W'_k.*

(iv) *The subvariety $\mathcal{B}_{\mathrm{x}}^{(k)} \setminus W_k$ of $\mathcal{B}_{\mathrm{x}}^{(k)}$ has codimension at least $2k$.*

Proof. (i) According to Corollary 2.6,(ii), θ is an isomorphism onto a smooth open subset of $\mathcal{X}$. As a result, W'_k is an open subset of $\mathcal{B}_{\mathrm{x}}^{(k)}$ and the map

$$U_{-} \times \mathfrak{b}_{\mathrm{reg}} \times \mathfrak{b}^{k-1} \longrightarrow W'_k, \qquad (g, x_1, \ldots, x_k) \longmapsto (g(x_1), \ldots, g(x_k), \overline{x_1}, \ldots, \overline{x_k})$$

is an isomorphism whose inverse is given by

$$W'_k \longrightarrow U_{-} \times \mathfrak{b}_{\mathrm{reg}} \times \mathfrak{b}^{k-1},$$

$$(x_1, \ldots, x_k) \longmapsto (\theta^{-1}(x_1, \overline{x_1})_1, \theta^{-1}(x_1, \overline{x_1})_1(x_1), \ldots, \theta^{-1}(x_1, \overline{x_1})_1(x_k))$$

with θ^{-1} the inverse of θ and $\theta^{-1}(x_1, \overline{x_1})_1$ the component of $\theta^{-1}(x_1, \overline{x_1})$ on U_{-}, whence the assertion since $U_{-} \times \mathfrak{b}_{\mathrm{reg}} \times \mathfrak{b}^{k-1}$ is smooth.

(ii) For $(x_1, \ldots, x_k)$ in $\mathfrak{b}^k$ and for $(a_{i,j}, 1 \leq i, j \leq k)$ in $\mathrm{GL}_k(\Bbbk)$,

$$\overline{\sum_{j=1}^{k} a_{i,j} x_j} = \sum_{j=1}^{k} a_{i,j} \overline{x_j}$$

so that $\iota_k(\mathfrak{b}^k)$ is invariant under the action of $\mathrm{GL}_k(\Bbbk)$ in $\mathfrak{g}^k \times \mathfrak{h}^k$ defined by

$$(a_{i,j}, 1 \leq i, j \leq k).(x_1, \ldots, x_k, y_1, \ldots, y_k) := (\sum_{j=1}^{k} a_{i,j} x_j, j = 1, \ldots, k, \sum_{j=1}^{k} a_{i,j} y_j, j = 1, \ldots, k),$$

whence the assertion since $\mathcal{B}_{\mathrm{x}}^{(k)} = G.\iota_k(\mathfrak{b}^k)$ and the actions of G and $\mathrm{GL}_k(\Bbbk)$ in $\mathfrak{g}^k \times \mathfrak{h}^k$ commute.

(iii) According to (i), $G.W'_k$ is a smooth open subset of $\mathcal{B}_{\mathrm{x}}^{(k)}$. Moreover, $G.W'_k$ is the subset of elements (x,y) such that the first component of x is regular. So, by (ii) and Lemma 1.9, $W_k = \mathrm{GL}_k(\Bbbk).(G.W'_k)$, whence the assertion.

(iv) According to Corollary 2.8,(i), $\mathcal{B}_{\mathrm{x}}^{(k)}$ is the image of $G \times_B \mathfrak{b}^k$ by the restriction γ_{x} of $\chi_{\mathrm{n}}^{(k)}$ to $G\times_B \mathfrak{b}^k$. Then $\mathcal{B}_{\mathrm{x}}^{(k)} \setminus W_k$ is contained in the image of $G\times_B (\mathfrak{b} \setminus \mathfrak{b}_{\mathrm{reg}})^k$ by γ_{x}. As a result, by Lemma 2.9,

$$\dim \mathcal{B}_{\mathrm{x}}^k \setminus W_k \leq n + k(\mathrm{b}_{\mathfrak{g}} - 2),$$

whence the assertion. □

Corollary 2.11. *The restriction of γ_{x} to $\gamma_{\mathrm{x}}^{-1}(W_k)$ is an isomorphism onto W_k. Moreover, $\gamma_{\mathrm{x}}^{-1}(W_k)$ is a big open subset of $G \times_B \mathfrak{b}^k$.*

Proof. Since the subset of Borel subalgebras containing a regular element is finite, the fibers of γ_{x} over the elements of W_k are finite. In particular, the restriction of γ_{x} to $\gamma_{\mathrm{x}}^{-1}(W_k)$ is a quasi finite surjective morphism onto W_k. So, by Zariski's Main Theorem [Mu88, §9], it is an isomorphism since W_k is smooth by Lemma 2.10,(iii).

Recall that $G \times_B \mathfrak{b}^k$ identifies with a closed subset of $G/B \times \mathfrak{g}^k$. For u in G/B and x in $\mathfrak{g}^k$ such that (u, x) is in $G \times_B \mathfrak{b}^k$, (u, x) is in $\gamma_{\mathrm{x}}^{-1}(W_k)$ if and only if $E_x \cap \mathfrak{g}_{\mathrm{reg}}$ is not empty. Denote by π the bundle projection of the vector bundle $G \times_B \mathfrak{b}^k$ over G/B. Let Σ be an irreducible component of $G \times_B \mathfrak{b}^k \setminus \gamma_{\mathrm{x}}^{-1}(W_k)$. For u in $\pi(\Sigma)$, set:

$$\Sigma_u := \{x \in \mathfrak{g}^k \mid (u, x) \in \Sigma\}.$$

Since W_k is a cone, for all u in $\pi(\Sigma)$, Σ_u is a closed cone of u^k, whence $\pi(\Sigma) \times \{0\} = \Sigma \cap G/B \times \{0\}$ so that $\pi(\Sigma)$ is a closed subset of G/B. Suppose that Σ has codimension 1 in $G \times_B \mathfrak{b}^k$. A contradiction is expected. Then $\pi(\Sigma)$ has codimension at most 1 in G/B. If $\pi(\Sigma)$ has codimension 1 in G/B, then for all u in $\pi(\Sigma)$, $\Sigma_u = u^k$. It is impossible since $u \cap \mathfrak{g}_{\mathrm{reg}}$ is not empty. As a result, for all u in a dense open subset of G/B, Σ_u is closed of codimension 1 in u^k. According to Lemma 2.9, $u \setminus \mathfrak{g}_{\mathrm{reg}}$ has codimension 2 in u and Σ_u is contained in $(u \setminus \mathfrak{g}_{\mathrm{reg}})^k$, whence the contradiction. □

2.5. For E a B-module, denote by $\mathcal{L}_0(E)$ the sheaf of local sections of the vector bundle $G \times_B E$ over G/B. Let Δ be the diagonal of $(G/B)^k$ and let $\mathcal{J}_\Delta$ be its ideal of definition in $\mathcal{O}_{(G/B)^k}$. The variety G/B identifies with Δ so that $\mathcal{O}_{(G/B)^k}/\mathcal{J}_\Delta$ is isomorphic to $\mathcal{O}_{G/B}$. For E a B^k-module, denote by $\mathcal{L}(E)$ the sheaf of local sections of the vector bundle $G^k \times_{B^k} E$ over $(G/B)^k$.

Lemma 2.12. *Let E be a B^k-module. Denote by $\overline{E}$ the B-module defined by the diagonal action of B on E. The short sequence of $\mathcal{O}_{(G/B)^k}$-modules*

$$0 \longrightarrow \mathcal{J}_\Delta \otimes_{\mathcal{O}_{G^k \times_{B^k} \mathfrak{b}^k}} \mathcal{L}(E) \longrightarrow \mathcal{L}(E) \longrightarrow \mathcal{L}_0(\overline{E}) \longrightarrow 0$$

is exact.

Proof. Since $\mathcal{L}(E)$ is a locally free $\mathcal{O}_{(G/B)^k}$-module, the short sequence of $\mathcal{O}_{(G/B)^k}$-modules

$$0 \longrightarrow \mathcal{J}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(E) \longrightarrow \mathcal{L}(E) \longrightarrow \mathcal{O}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(E) \longrightarrow 0$$

is exact, whence the assertion since $\mathcal{O}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(E)$ is isomorphic to $\mathcal{L}_0(\overline{E})$. □

From Lemma 2.12 results a canonical morphism

$$\mathrm{H}^0((G/B)^k, \mathcal{L}(E)) \longrightarrow \mathrm{H}^0(G/B, \mathcal{L}_0(\overline{E}))$$

for all B^k-module E. According to the identification of $\mathfrak{g}$ and $\mathfrak{g}^*$ by $\langle .,. \rangle$, the duals of $\mathfrak{b}$ and $\mathfrak{u}$ identify with $\mathfrak{b}_-$ and $\mathfrak{u}_-$ respectively so that $\mathfrak{b}_-$ and $\mathfrak{u}_-$ are B-modules.

Lemma 2.13. (i) *The algebra* $\Bbbk[\mathcal{B}_{\rm n}^{(k)}]$ *is equal to* $\mathrm{H}^0(G/B, \mathcal{L}_0(\overline{\mathrm{S}(\mathfrak{b}_-^k}))).$

(ii) *The algebra* $\Bbbk[\mathcal{N}_{\rm n}^{(k)}]$ *is equal to* $\mathrm{H}^0(G/B, \mathcal{L}_0(\overline{\mathrm{S}(\mathfrak{u}_-^k}))).$

(iii) *The algebra* $\Bbbk[\mathcal{B}_{\rm x}^{(k)}]$ *is the image of the morphism*

$$\mathrm{H}^0((G/B)^k, \mathcal{L}(\mathrm{S}(\mathfrak{b}_-^k))) \longrightarrow \mathrm{H}^0(G/B, \mathcal{L}_0(\overline{\mathrm{S}(\mathfrak{b}_-^k})))\ .$$

(iv) *The algebra* $\Bbbk[\mathcal{N}^{(k)}]$ *is the image of the morphism*

$$\mathrm{H}^0((G/B)^k, \mathcal{L}(\mathrm{S}(\mathfrak{u}_-^k))) \longrightarrow \mathrm{H}^0(G/B, \mathcal{L}_0(\overline{\mathrm{S}(\mathfrak{u}_-^k})))\ .$$

Proof. (i) Since $G \times_B \mathfrak{b}^k$ is a desingularization of the normal variety $\mathcal{B}_{\rm n}^{(k)}$, $\Bbbk[\mathcal{B}_{\rm n}^{(k)}]$ is the space of global sections of $\mathcal{O}_{G\times_B\mathfrak{b}^k}$ by Lemma 1.4. Let π be the bundle projection of the fiber bundle $G\times_B \mathfrak{b}^k$. Since $\mathrm{S}(\mathfrak{b}_-^k)$ is the space of polynomial functions on $\mathfrak{b}^k$,

$$\pi_*(\mathcal{O}_{G\times_B\mathfrak{b}^k}) = \mathcal{L}_0(\overline{\mathrm{S}(\mathfrak{b}_-^k})),$$

whence the assertion.

(ii) By Lemma 2.1,(ii), $G\times_B \mathfrak{u}^k$ is a desingularization of $\mathcal{N}_{\rm n}^{(k)}$ so that $\Bbbk[\mathcal{N}_{\rm n}^{(k)}]$ is the space of global sections of $\mathcal{O}_{G\times_B\mathfrak{u}^k}$ by Lemma 1.4. Denoting by π_0 the bundle projection of $G \times_B \mathfrak{u}^k$,

$$\pi_{0*}(\mathcal{O}_{G\times_B\mathfrak{u}^k}) = \mathcal{L}_0(\overline{\mathrm{S}(\mathfrak{u}_-^k})),$$

whence the assertion.

(iii) Since $G^k \times_{B^k} \mathfrak{b}^k$ is isomorphic to $(G\times_B \mathfrak{b})^k$,

$$\mathrm{H}^0((G/B)^k, \mathcal{O}_{G^k\times_{B^k}\mathfrak{b}^k}) = \mathrm{H}^0(G/B, \mathcal{O}_{G\times_B\mathfrak{b}})^{\otimes k}.$$

By (i),

$$H^0(G/B, \mathcal{O}_{G\times_B\mathfrak{b}}) = \mathrm{H}^0(G/B, \mathcal{L}(\mathrm{S}(\mathfrak{b}_-)) = \Bbbk[\mathcal{X}]$$

since $G\times_B \mathfrak{b}$ is a desingularization of $\mathcal{X}$ by Lemma 2.1,(i) and (ii), whence

$$\mathrm{H}^0((G/B)^k, \mathcal{L}(\mathrm{S}(\mathfrak{b}_-^k)) = \Bbbk[\mathcal{X}^k].$$

By definition, $\mathcal{B}_{\rm x}^{(k)}$ is a closed subvariety of $\mathcal{X}^k$. According to Corollary 2.8, $\Bbbk[\mathcal{B}^{(k)}]$ is a subalgebra of $\Bbbk[\mathcal{B}_{\rm x}^{(k)}]$ having the same fraction field and $\Bbbk[\mathcal{B}_{\rm x}^{(k)}]$ is a finite extension of $\Bbbk[\mathcal{B}^{(k)}]$. Hence $\Bbbk[\mathcal{B}_{\rm x}^{(k)}]$ is a subalgebra of $\Bbbk[\mathcal{B}_{\rm n}^{(k)}]$. For φ in $\Bbbk[\mathcal{B}_{\rm x}^{(k)}]$, φ is the restriction to $\mathcal{B}_{\rm x}^{(k)}$ of an element ψ of $\Bbbk[\mathcal{X}^{(k)}]$. As mentioned above, ψ is a global section of $\mathcal{L}(\mathfrak{b}_-^k)$. Denoting by $\overline{\psi}$ its restriction to the diagonal of $(G/B)^k$, $\overline{\psi}$ is a global section of $\mathcal{L}_0(\overline{\mathrm{S}(\mathfrak{b}_-^k}))$ so that $\overline{\psi}$ is in $\Bbbk[\mathcal{B}_{\rm n}^{(k)}]$. Moreover, for all smooth point x of $\mathcal{B}_{\rm x}^{(k)}$, $\overline{\psi}(x) = \varphi(x)$. Hence φ is in the image of the morphism

$$\mathrm{H}^0((G/B)^k, \mathcal{L}(\mathrm{S}(\mathfrak{b}_-^k))) \longrightarrow \mathrm{H}^0(G/B, \mathcal{L}_0(\overline{\mathrm{S}(\mathfrak{b}_-^k})))\ .$$

Conversely, for φ image of ψ in $\mathrm{H}^0((G/B)^k, \mathcal{L}(\mathrm{S}(\mathfrak{b}_-^k))$ by this morphism, ψ is in $\Bbbk[\mathcal{X}^k]$ and $\varphi(x) = \psi(x)$ for all smooth point x of $\mathcal{B}_{\rm x}^{(k)}$ so that φ is the restriction of ψ to $\mathcal{B}_{\rm x}^{(k)}$.

(iv) Let φ be in $\Bbbk[\mathcal{N}^{(k)}]$. Since $\mathcal{N}^{(k)}$ is a closed subvariety of $\mathfrak{N}_{\mathfrak{g}}^k$, φ is the restriction to $\mathcal{N}^{(k)}$ of an element ψ of $\Bbbk[\mathfrak{N}_{\mathfrak{g}}^k]$. As mentioned above, ψ is a global section of $\mathcal{L}(\mathrm{S}(\mathfrak{u}_-^k))$. Denoting by $\overline{\psi}$ the restriction of ψ to the diagonal of $(G/B)^k$, $\overline{\psi}$ is a global section of $\mathcal{L}_0(\overline{\mathrm{S}(\mathfrak{u}_-^k}))$ so that $\overline{\psi}$ is in $\Bbbk[\mathcal{N}_{\rm n}^{(k)}]$. Moreover, for all smooth point x of $\mathcal{N}^{(k)}$, $\overline{\psi}(x) = \varphi(x)$. Hence φ is in the image of the morphism

$$\mathrm{H}^0((G/B)^k, \mathcal{L}(\mathrm{S}(\mathfrak{u}_-^k))) \longrightarrow \mathrm{H}^0(G/B, \mathcal{L}_0(\overline{\mathrm{S}(\mathfrak{u}_-^k})))\ .$$

Conversely, for φ image of ψ in $\mathrm{H}^0((G/B)^k, \mathcal{L}(\mathrm{S}(\mathfrak{u}_-^k))$ by this morphism, ψ is in $\Bbbk[\mathfrak{N}_\mathfrak{g}^k]$ and $\varphi(x) = \psi(x)$ for all smooth point x of $\mathcal{N}^{(k)}$ so that φ is the restriction of ψ to $\mathcal{N}^{(k)}$. □

Let $x_1, \ldots, x_{k\ell}$ be a basis of $\mathfrak{h}^k$ verifying the following conditions for $j = 1, \ldots, k$:

(1) $x_{j,l} = 0$ for $l < (j-1)\ell$ and $l > j\ell$

(2) $x_{j,j\ell} = h$,

(3) $x_{j,(j-1)\ell+1}, \ldots, x_{j,j\ell-1}$ is a basis of the orthogonal complement to h in $\mathfrak{h}$,

with $x_{j,l}$ the component of x_l on the j-th factor $\mathfrak{h}$. Set:

$$E_0 := \{0\}, \qquad F_0 := \mathfrak{b}_-^k, \qquad E_i := \mathrm{span}(\{x_1, \ldots, x_i\}), \qquad F_i := \mathfrak{b}_-^k/E_i$$

for $i = 1, \ldots, k\ell$. In the B-module $\mathfrak{b}_-$, $\mathfrak{h}$ is the subspace of invariant elements and $\mathfrak{u}_-$ is the quotient of $\mathfrak{b}_-$ by $\mathfrak{h}$. So for $i = 0, \ldots, k\ell$, F_i is a B^k-module. As a matter of fact, because of the choice of the basis $x_1, \ldots, x_{k\ell}$, $F_i = F_{i,1} \times \cdots \times F_{i,k}$ where $F_{i,1}, \ldots, F_{i,k}$ are B-modules quotient of $\mathfrak{b}_-$ and the action of B^k on F_i is the product action. For $i = 0, \ldots, k\ell$, set:

$$A_i := \mathrm{H}^0(G/B, \mathcal{L}_0(\overline{\mathrm{S}(F_i)})) \quad \text{and} \quad C_i := \mathrm{H}^0((G/B)^k, \mathcal{L}(\mathrm{S}(F_i))).$$

Denote by B_i the image of C_i by the restriction map to the diagonal of $(G/B)^k$. Then A_i, B_i, C_i are integral graded algebras. Moreover, by Lemma 1.4, A_i and C_i are normal as spaces of global sections of structural sheaves of vector bundles over G/B and $(G/B)^k$. For $i < k\ell$, the B^k-module F_{i+1} is a quotient of the B^k-module F_i so that $\mathrm{S}(F_{i+1})$ is a quotient of $\mathrm{S}(F_i)$ as a B^k-algebra and $\mathcal{L}(\mathrm{S}(F_{i+1}))$ and $\mathcal{L}_0(\overline{\mathrm{S}(F_{i+1})})$ are quotients of $\mathcal{L}(\mathrm{S}(F_i))$ and $\mathcal{L}_0(\overline{\mathrm{S}(F_i)})$ respectively, whence morphisms of algebras

$$C_i \xrightarrow{\ \nu_i\ } C_{i+1} \quad \text{and} \quad A_i \xrightarrow{\ \nu_{i,0}\ } A_{i+1} .$$

For $i = 0, \ldots, k\ell - 1$ and m positive integer, denote again by x_{i+1} the image of x_{i+1} in F_i by the quotient morphism and by $J_{m,i}$ the ideal of $\mathrm{S}(F_i)$ generated by x_{i+1}^m. As x_{i+1} is a fixed point of the B^k-module F_i, $J_{m,i}$ is a B^k-submodule of $\mathrm{S}(F_i)$.

Lemma 2.14. *Let $i = 0, \ldots, k\ell - 1$ and m a positive integer.*

(i) *The algebra $\Bbbk[x_{i+1}]$ is canonically embedded in A_i and C_i.*

(ii) *The space $\mathrm{H}^0(G/B, \mathcal{L}_0(\overline{J_{m,i}}))$ is the ideal of A_i generated by x_{i+1}^m and the image of the canonical morphism*

$$\mathrm{H}^0(G/B, \mathcal{L}_0(\overline{J_{m,i}})) \longrightarrow \mathrm{H}^0(G/B, \mathcal{L}_0(\overline{\mathrm{S}(F_{i+1}) \otimes_\Bbbk \Bbbk x_{i+1}^m}))$$

is equal to $\nu_{i,0}(A_i) \otimes_\Bbbk \Bbbk x_{i+1}^m$.

(iii) *The space $\mathrm{H}^0((G/B)^k, \mathcal{L}(J_{m,i}))$ is the ideal of C_i generated by x_{i+1}^m and the image of the canonical morphism*

$$\mathrm{H}((G/B)^k, \mathcal{L}(J_{m,i})) \longrightarrow \mathrm{H}((G/B)^k, \mathcal{L}(\mathrm{S}(F_{i+1}) \otimes_\Bbbk \Bbbk x_{i+1}^m))$$

is equal to $C_{i+1} \otimes_\Bbbk \Bbbk x_{i+1}^m$.

(iv) *Let $v_1, \ldots, v_l$ be in A_i such that $\nu_{i,0}(v_1), \ldots, \nu_{i,0}(v_l)$ are linearly free over $\Bbbk$. Then $v_1, \ldots, v_l$ are linearly free over $\Bbbk[x_{i+1}]$.*

(v) *Let $w_1, \ldots, w_l$ be in C_i such that $\nu_i(w_1), \ldots, \nu_i(w_l)$ are linearly free over $\Bbbk$. Then $w_1, \ldots, w_l$ are linearly free over $\Bbbk[x_{i+1}]$.*

Proof. (i) Since x_{i+1} is a fixed point of the B^k-module $\mathrm{S}(F_i)$, $\mathcal{L}(\Bbbk[x_{i+1}])$ and $\mathcal{L}_0(\overline{\Bbbk[x_{i+1}]})$ are submodules of $\mathcal{L}(\mathrm{S}(F_i))$ and $\mathcal{L}_0(\overline{\mathrm{S}(F_i)})$ respectively. Moreover they are isomorphic to $\mathcal{O}_{(G/B)^k}\otimes_{\Bbbk}\Bbbk[x_{i+1}]$ $\mathcal{O}_{G/B}\otimes_{\Bbbk}\Bbbk[x_{i+1}]$ respectively, whence the assertion.

(ii) Since F_{i+1} is the quotient of F_i by $\Bbbk x_{i+1}$, we have the exact sequence of B^k-modules,

$$0 \longrightarrow J_{m+1,i} \longrightarrow J_{m,i} \longrightarrow \mathrm{S}(F_{i+1})\otimes_{\Bbbk}\Bbbk x_{i+1}^m \longrightarrow 0,$$

whence the exact sequence of $\mathcal{O}_{G/B}$-modules,

$$0 \longrightarrow \mathcal{L}_0(\overline{J_{m+1,i}}) \longrightarrow \mathcal{L}_0(\overline{J_{m,i}}) \longrightarrow \mathcal{L}_0(\overline{\mathrm{S}(F_{i+1})\otimes_{\Bbbk}\Bbbk x_{i+1}^m}) \longrightarrow 0,$$

and whence the canonical morphism

$$\mathrm{H}^0(G/B,\mathcal{L}_0(\overline{J_{m,i}})) \longrightarrow \mathrm{H}^0(G/B,\mathcal{L}_0(\overline{\mathrm{S}(F_{i+1})\otimes_{\Bbbk}\Bbbk x_{i+1}^m}))\ .$$

In particular, $\nu_{i,0}(A_i)\otimes_{\Bbbk}\Bbbk x_{i+1}^m$ is contained in its image since the image of ax_{i+1}^m is equal to $\nu_{i,0}(a)\otimes x_{i+1}^m$ for all a in A_i.

Let a be in $\mathrm{H}^0(G/B,\mathcal{L}_0(\overline{J_{m,i}}))$. Let $O_1,\dots,O_l$ be a cover of G/B by affine trivialization open subsets of the vector bundle $G\times_B \mathrm{S}(F_i)$. For $j=1,\dots,l$, denoting by Φ_j a trivialization over O_j, we have a commutative diagram:

$$\begin{tikzcd} \pi_i^{-1}(O_j) \arrow[r, "\Phi_j"] \arrow[dr, "\pi_i"'] & O_j\times \mathrm{S}(F_i) \arrow[d, "\mathrm{pr}_1"] \\ & O_j \end{tikzcd}$$

with π_i the bundle projection. Since x_{i+1} is invariant under B, for φ local section of $\mathcal{L}_0(\overline{\mathrm{S}(F_i)})$ above O_j, $\Phi_j(x_{i+1}^m\varphi) = x_{i+1}^m\Phi_j(\varphi)$, whence

$$\Phi_{j_*}(\Gamma(O_j,\mathcal{L}_0(\overline{J_{m,i}})) = \Bbbk[O_j]\otimes_{\Bbbk}\mathrm{S}(F_i)x_{i+1}^m.$$

As a result, for some local section a_j above O_j of $\mathcal{L}_0(\overline{\mathrm{S}(F_i)})$, $a = x_{i+1}^m a_j$. Moreover, a_j is uniquely defined by this equality. Then for all j, j', a_j and $a_{j'}$ have the same restriction to $O_j\cap O_{j'}$. Denoting by a' the global section of $\mathcal{L}_0(\overline{\mathrm{S}(F_i)})$ extending $a_1,\dots,a_l$, $a = a'x_{i+1}^m$, whence the assertion.

(iii) According to [He76, Theorem B and Corollary], for a B-module quotient V of $\mathfrak{b}_-$ having $\mathfrak{u}_-$ as quotient, $\mathrm{H}^1(G/B,\mathcal{L}_0(V)) = 0$. Hence C_{i+1} is the image of ν_i. From the exact sequence of $(G/B)^k$-modules

$$0 \longrightarrow J_{m+1,i} \longrightarrow J_{m,i} \longrightarrow \mathrm{S}(F_{i+1})\otimes_{\Bbbk}\Bbbk x_{i+1}^m \longrightarrow 0,$$

we deduce the exact sequence of $\mathcal{O}_{(G/B)^k}$-modules,

$$0 \longrightarrow \mathcal{L}(J_{m+1,i}) \longrightarrow \mathcal{L}(J_{m,i}) \longrightarrow \mathcal{L}(\mathrm{S}(F_{i+1})\otimes_{\Bbbk}\Bbbk x_{i+1}^m) \longrightarrow 0,$$

and whence the canonical morphism

$$\mathrm{H}^0((G/B)^k,\mathcal{L}(J_{m,i})) \longrightarrow \mathrm{H}^0((G/B)^k,\mathcal{L}(\mathrm{S}(F_{i+1})\otimes_{\Bbbk}\Bbbk x_{i+1}^m))\ .$$

In particular, $C_{i+1}\otimes_{\Bbbk}\Bbbk x_{i+1}^m$ is its image since the image of ax_{i+1}^m is equal to $\nu_i(a)\otimes x_{i+1}^m$ for all a in C_i and C_{i+1} is the image of ν_i.

Let a be in $\mathrm{H}^0(G/B,\mathcal{L}(J_{m,i}))$. Prove by induction on l that for some a_l in C_i, $a - a_l x_{i+1}^m$ is in $\mathrm{H}^0(G/B,\mathcal{L}(J_{m+l,i}))$. It is true for $l=0$. Suppose that it is true for l. By the above result for $m+l$, for some a_l' in C_i, $a - a_l x_{i+1}^m - a_l' x_{i+1}^{m+l}$ is in $\mathrm{H}^0(G/B,\mathcal{L}(J_{m+l+1,i}))$, whence the statement. As C_i is a graded algebra and $\mathrm{H}^0(G/B, \mathcal{L}(J_{m+l,i}))$ is a graded subspace whose elements have degree at least $m+l$, for l sufficiently big, $a = a_l x_{i+1}^m$, whence the assertion.

(iv) Suppose that $v_1, \ldots, v_l$ is not linearly free over $\Bbbk[x_{i+1}]$. A contradiction is expected. Let $a_1, \ldots, a_l$ be in $\Bbbk[x_{i+1}]$ such that $a_i \neq 0$ for some i and

$$a_1 v_1 + \cdots + a_l v_l = 0.$$

Suppose that m is the biggest integer such that x_{i+1}^m divides $a_1, \ldots, a_l$ in $\Bbbk[x_{i+1}]$. For $j = 1, \ldots, l$, denote by c_j the element of $\Bbbk$ such that x_{i+1}^{m+1} divides $a_j - c_j x_{i+1}^m$. Then by (ii),

$$c_1 \nu_{i,0}(v_1) + \cdots + c_l \nu_{i,0}(v_l) = 0,$$

whence a contradiction by the maximality of m.

(v) Suppose that $w_1, \ldots, w_l$ is not linearly free over $\Bbbk[x_{i+1}]$. A contradiction is expected. Let $a_1, \ldots, a_l$ be in $\Bbbk[x_{i+1}]$ such that $a_i \neq 0$ for some i and

$$a_1 w_1 + \cdots + a_l w_l = 0.$$

Suppose that m is the biggest integer such that x_{i+1}^m divides $a_1, \ldots, a_l$ in $\Bbbk[x_{i+1}]$. For $j = 1, \ldots, l$, denote by c_j the element of $\Bbbk$ such that x_{i+1}^{m+1} divides $a_j - c_j x_{i+1}^m$. Then by (iii),

$$c_1 \nu_i(w_1) + \cdots + c_l \nu_i(w_l) = 0,$$

whence a contradiction by the maximality of m. □

Corollary 2.15. *Let $i = 0, \ldots, k\ell - 1$.*

(i) *The algebra A_i is a free extension of $\Bbbk[x_{i+1}]$ and $\nu_{i,0}(A_i)$ is the quotient of A_i by the ideal generated by x_{i+1}.*

(ii) *The algebra C_i is a free extension of $\Bbbk[x_{i+1}]$ and C_{i+1} is the quotient of C_i by the ideal generated by x_{i+1}.*

(iii) *The algebra B_i is a free extension of $\Bbbk[x_{i+1}]$ and B_{i+1} is the quotient of B_i by the ideal generated by x_{i+1}.*

Proof. (i) Let K_0 be a $\Bbbk$-subspace of A_i such that the restriction of $\nu_{i,0}$ to K_0 is an isomorphism onto the $\Bbbk$-space $\nu_{i,0}(A_i)$. Prove by induction on m the equality

$$A_i = K_0 \Bbbk[x_{i+1}] + \mathrm{H}^0(G/B, \mathcal{L}_0(\overline{J_{m,i}})).$$

The equality is true for $m = 0$. Suppose that it is true for m. Let a be in $\mathrm{H}^0(G/B, \mathcal{L}_0(\overline{J_{m,i}}))$. By Lemma 2.14,(ii), for some b in A_i, $a - b x_{i+1}^m$ is in $\mathrm{H}^0(G/B, \mathcal{L}_0(\overline{J_{m+1,i}}))$, whence the equality. Since A_i is graded with x_{i+1} having degree 1, $A_i = K_0 \Bbbk[x_{i+1}]$. So A_i is a free $\Bbbk[x_{i+1}]$-module by Lemma 2.14,(iv). Again by Lemma 2.14,(ii), $\nu_{i,0}(A_i)$ is the quotient of A_i by the ideal generated by x_{i+1}.

(ii) Let K be a $\Bbbk$-subspace of C_i such that the restriction of ν_i to K is an isomorphism onto the $\Bbbk$-space C_{i+1}. Prove by induction on m the equality

$$C_i = K \Bbbk[x_{i+1}] + \mathrm{H}^0((G/B)^k, \mathcal{L}(J_{m,i})).$$

The equality is true for $m = 0$. Suppose that it is true for m. Let a be in $\mathrm{H}^0((G/B)^k, \mathcal{L}(J_{m,i})$. By Lemma 2.14,(iii), for some b in C_i, $a - b x_{i+1}^m$ is in $\mathrm{H}^0((G/B)^k, \mathcal{L}(J_{m+1,i}))$, whence the equality. Since C_i is graded with x_{i+1} having degree 1, $C_i = K \Bbbk[x_{i+1}]$. So C_i is a free $\Bbbk[x_{i+1}]$-module by Lemma 2.14,(v). Again by Lemma 2.14,(iii), C_{i+1} is the quotient of C_i by the ideal generated by x_{i+1}.

(iii) We have the commutative diagram

$$\begin{tikzcd} C_i \arrow[r, "\nu_i"] \arrow[d] & C_{i+1} \arrow[d] \\ B_i \arrow[r, "\nu'_{i,0}"] & B_{i+1} \end{tikzcd}$$

where the vertical arrows are the restriction morphisms to the diagonal of $(G/B)^k$ and $\nu'_{i,0}$ is the restriction of $\nu_{i,0}$ to B_i. In particular $\nu'_{i,0}$ is surjective since so is ν_i. Let K'_0 be a $\Bbbk$-subspace of B_i such that the restriction of $\nu'_{i,0}$ to K'_0 is an isomorphism onto the $\Bbbk$-space B_{i+1}. For m positive integer, denote by $\mathfrak{J}_{m,i}$ the image of the restriction morphism to the diagonal of $(G/B)^k$,

$$\mathrm{H}^0((G/B)^k, \mathcal{L}(J_{m,i})) \longrightarrow B_i \ .$$

Prove by induction on m the equality

$$B_i = K'_0\Bbbk[x_{i+1}] + \mathfrak{J}_{m,i}.$$

The equality is true for $m = 0$. Suppose that it is true for m. Let a be in $\mathfrak{J}_{m,i}$. Then a is the image of an element a' of $\mathrm{H}^0((G/B)^k, \mathcal{L}(J_{m,i}))$. By Lemma 2.14,(iii), for some b' in C_i, $a' - b'x_{i+1}^m$ is in $\mathrm{H}^0((G/B)^k, \mathcal{L}(J_{m+1,i}))$. Denoting by b the image of b' in B_i, $a - bx_{i+1}^m$ is in $\mathfrak{J}_{m+1,i}$, whence the equality. Since B_i is graded with x_{i+1} having degree 1, $B_i = K'_0\Bbbk[x_{i+1}]$. So B_i is a free $\Bbbk[x_{i+1}]$-module by Lemma 2.14,(iv).

Since B_{i+1} is contained in $\nu_{i,0}(A)$, K'_0 can be chosen contained in K_0. Let $v_l, l \in L$ be a basis of K_0 such that $v_l, l \in L'$ is a basis of K'_0 for some subset L' of L. Let a be in the kernel of $\nu'_{i,0}$. Then a is in the kernel of $\nu_{i,0}$ so that $a = bx_{i+1}$ for some b in A_i by Lemma 2.14,(ii). By (i) and the freeness of the extension B_i of $\Bbbk[x_{i+1}]$,

$$b = \sum_{l\in L} v_l p_l \quad \text{and} \quad a = \sum_{l\in L'} v_l q_l$$

with $p_l, l \in L$ and $q_l, l \in L'$ in $\Bbbk[x_{i+1}]$ with finite supports. Then $q_l = p_l x_{i+1}$ for all l in L' and $p_l = 0$ for l in $L \setminus L'$ so that a is in $B_i x_{i+1}$, whence the assertion. □

For $j = 0, \ldots, k\ell$, set $F_j^* := \mathrm{Specm}(\mathrm{S}(F_j))$. By definition, F_j^* is the subspace of elements $(y_1, \ldots, y_k)$ of $\mathfrak{b}^k$ such that for $m = 1, \ldots, k$ and $l = 1, \ldots, j$, $\langle y_m, x_{m,l}\rangle = 0$.

Lemma 2.16. *Let $i = 0, \ldots, k\ell - 1$. Denote by T the annihilator of x_{i+1} in the A_i-module* $\mathrm{H}^1(G/B, \mathcal{L}_0(\overline{\mathrm{S}(F_i)}))$.

(i) *The algebra A_i is the integral closure of B_i in its fraction field.*

(ii) *There is a well defined morphism $\mu : T \longrightarrow A_{i+1}$ of A_i-modules.*

(iii) *The A_i-module A_{i+1} is the direct sum of $\mu(T)$ and $\nu_{i,0}(A_i)$.*

Proof. Since A_i is the space of global sections of $\mathcal{L}_0(\overline{\mathrm{S}(F_i)})$, for all integer m, $\mathrm{H}^m(G/B, \mathcal{L}_0(\overline{\mathrm{S}(F_i)}))$ is a A_i-module.

(i) According to the proof of Lemma 2.1, $\Omega_{\mathfrak{g}}^{(k)}$ is an open subset of $\mathfrak{g}^k$ such that for all x in $\Omega_{\mathfrak{g}}^{(k)} \cap \mathcal{B}^{(k)}$, there exists only one Borel subalgebra of $\mathfrak{g}$ containing E_x. By definition, F_i^* is the subspace of elements $(y_1, \ldots, y_k)$ of $\mathfrak{b}^k$ such that for $j = 1, \ldots, k$ and $l = 1, \ldots, i$, $\langle y_j, x_{l,j}\rangle = 0$. By Lemma 1.7, $G.F_i^*$ is a closed subset of $\mathfrak{g}^k$ and the morphism $G \times_B F_i^* \longrightarrow G.F_i^*$ is projective. By Conditions (1), (2), (3), for some $y_2, \ldots, y_{k-1}$ in $\mathfrak{b}$, $(e, y_2, \ldots, y_{k-1}, h)$ is in F_i^*. Hence $\Omega_{\mathfrak{g}}^{(k)} \cap G.F_i^*$

is a dense open subset of $G.F_i^*$ and the above morphism is birational. So, by Lemma 1.4, A_i is the integral closure of $\Bbbk[G.F_i^*]$ in its fraction field.

By Lemma 1.7, $G^k.F_i^*$ is a closed subset of $\mathfrak{g}^k$ containing $G.F_i^*$. Then for all φ in $\Bbbk[G.F_i^*]$, φ is the restriction to $G.F_i^*$ of an element ψ of $\Bbbk[G^k.F_i^*]$ so that ψ is a global section of $\mathcal{L}(\mathrm{S}(F_i^*))$. Denoting by $\overline{\psi}$ the restriction of ψ to the diagonal of $(G/B)^k$, $\varphi = \overline{\psi}$. Hence $\Bbbk[G.F_i^*]$ is contained in B_i so that A_i is the integral closure of B_i in its fraction field.

(ii) Let $O := O_1, \dots, O_m$ be a cover of G/B by open subsets isomorphic to the affine space of dimension n so that O_i is a trivialization open subset of the vector bundles $G \times_B \overline{\mathrm{S}(F_i)}$ and $G \times_B \overline{\mathrm{S}(F_{i+1})}$. Denote by $\mathcal{Z}^1$ the space of cocycles of degree 1 of the complex $C^\bullet$ of Čech cohomology of O with values in $\mathcal{L}_0(\overline{\mathrm{S}(F_i)})$.

Let $\overline{a}$ be in T and a a representative of $\overline{a}$ in $\mathcal{Z}^1$. Since $\overline{a}$ is in T, $x_{i+1}a$ is the boundary of an element b of C^0. For $l = 1, \dots, m$, denote by b_l the component of b in $\Gamma(O_l, \mathcal{L}_0(\overline{\mathrm{S}(F_i)}))$ and by $\widetilde{b_l}$ its image in $\Gamma(O_l, \mathcal{L}_0(\overline{\mathrm{S}(F_{i+1})}))$ by the quotient morphism. Then for $1 \leq l, l' \leq m$, $\widetilde{b_l}$ and $\widetilde{b_{l'}}$ have the same restriction to $O_l \cap O_{l'}$ so that $\widetilde{b_l}$ is the restriction to O_l of an element $\widetilde{b}$ of A_{i+1}. If a' is a representative of $\overline{a}$ in $\mathcal{Z}^1$, $a' - a$ is the boundary of an element b' in C^0 and $x_{i+1}a$ is the boundary of $b + x_{i+1}b'$. Hence $\widetilde{b}$ only depends on $\overline{a}$, whence a well defined map $T \longrightarrow A_{i+1}$. It is clearly a morphism of A_i-modules.

(iii) From the exact sequence of $\mathcal{O}_{G/B}$-modules

$$0 \longrightarrow \mathcal{L}_0(\overline{x_{i+1}\mathrm{S}(F_i)}) \longrightarrow \mathcal{L}_0(\overline{\mathrm{S}(F_i)}) \longrightarrow \mathcal{L}_0(\overline{\mathrm{S}(F_{i+1})}) \longrightarrow 0$$

we deduce the long exact sequence of cohomology

$$\cdots \longrightarrow A_i \longrightarrow A_{i+1} \longrightarrow x_{i+1}\mathrm{H}^1(G/B, \mathcal{L}_0(\overline{\mathrm{S}(F_i)})) \longrightarrow \mathrm{H}^1(G/B, \mathcal{L}_0(\overline{\mathrm{S}(F_i)})) \longrightarrow \cdots$$

since x_{i+1} is a global section of $\mathcal{L}_0(\overline{\mathrm{S}(F_i)})$ by Lemma 2.14,(i). Since the $\mathcal{O}_{G/B}$-modules $\mathcal{L}_0(\overline{x_{i+1}\mathrm{S}(F_i)})$ and $\mathcal{L}_0(\overline{\mathrm{S}(F_i)})$ are isomorphic, we have an isomorphism

$$\mathrm{H}^1(G/B, \mathcal{L}_0(\overline{x_{i+1}\mathrm{S}(F_i)})) \longrightarrow \mathrm{H}^1(G/B, \mathcal{L}_0(\overline{\mathrm{S}(F_i)}))$$

and the image of the kernel of the arrow

$$x_{i+1}\mathrm{H}^1(G/B, \mathcal{L}_0(\overline{\mathrm{S}(F_i)})) \longrightarrow \mathrm{H}^1(G/B, \mathcal{L}_0(\overline{\mathrm{S}(F_i)}))$$

by this isomorphism is equal to T, whence an exact sequence

$$A_i \longrightarrow A_{i+1} \longrightarrow T \longrightarrow 0.$$

By (ii), from the definition of the arrow

$$\mathrm{H}^0(G/B, \mathcal{L}_0(\overline{\mathrm{S}(F_{i+1})})) \longrightarrow x_{i+1}\mathrm{H}^1(G/B, \mathcal{L}_0(\overline{\mathrm{S}(F_i)}))$$

we deduce that for a in T, the image of $\mu(a)$ is equal to a. Hence A_{i+1} is the direct sum of $\mu(T)$ and $\nu_{i,0}(A_i)$. □

Corollary 2.17. *For $i = 0, \dots, k\ell - 1$, $\nu_{i,0}(A_i)$ is equal to A_{i+1}.*

Proof. According to Lemma 2.16, A_{i+1} is the direct sum of $\mu(T)$ and $\nu_{i,0}(A_i)$. Let a be in $\mu(T)$. By Lemma 2.16,(i) A_{i+1} and $\nu_{i,0}(A_i)$ have the same fraction field since B_{i+1} is contained in $\nu_{i,0}(A_i)$ by the proof of Corollary 2.15,(iii). So for some b in $\nu_{i,0}(A_i)$, ba is in $\nu_{i,0}(A_i)$, whence $ba = 0$ by Lemma 2.16,(ii) and (iii). As a result, $\mu(T) = \{0\}$ and $\nu_{i,0}(A_i) = A_{i+1}$. □

Proposition 2.18. (i) *The algebra* $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ *is a free extension of* $\mathrm{S}(\mathfrak{h}^k)$ *and* $\Bbbk[\mathcal{N}_{\mathrm{n}}^{(k)}]$ *is the quotient of* $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ *by the ideal generated by* $\mathrm{S}_+(\mathfrak{h}^k)$.

(ii) *The algebra* $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]$ *is a free extension of* $\mathrm{S}(\mathfrak{h}^k)$ *and* $\Bbbk[\mathcal{N}^{(k)}]$ *is the quotient of* $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]$ *by the ideal generated by* $\mathrm{S}_+(\mathfrak{h}^k)$.

Proof. (i) According to Lemma 2.13,(i) and (ii), $A_0 = \Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ and $A_{k\ell} = \Bbbk[\mathcal{N}_{\mathrm{n}}^{(k)}]$. Moreover, $\mathrm{S}(\mathfrak{h}^k) = \Bbbk[x_1, \dots, x_{k\ell}]$ by definition. So the assertion will result from the following claim:

Claim 2.19. For $i = 1, \dots, k\ell$, A_0 is a free extension of $\Bbbk[x_1, \dots, x_i]$ and A_i is the quotient of A_0 by the ideal generated by $x_1, \dots, x_i$.

Prove the claim by induction on i. According to Corollary 2.15,(i) and Corollary 2.17, the claim is true for $i = 1$. Suppose that it is true for i. By Corollary 2.15,(i) and Corollary 2.17, A_{i+1} is the quotient of A_i by the ideal generated by x_{i+1}. So by induction hypothesis, A_{i+1} is the quotient of A_0 by the ideal generated by $x_1, \dots, x_{i+1}$. For $j = 1, \dots, i+1$, denote by μ_j the quotient morphism $A_0 \longrightarrow A_j$. Let K_{i+1} be a $\Bbbk$-subspace of A_0 such that the restriction of μ_{i+1} to K_{i+1} is a $\Bbbk$-linear isomorphism onto A_{i+1}. Then $A_0 = K_{i+1} + A_0 x_1 + \cdots + A_0 x_{i+1}$. So by induction on m,

$$A_0 = K_{i+1}\Bbbk[x_1, \dots, x_{i+1}] + A_0 \mathfrak{I}_m$$

with $\mathfrak{I}_m$ the ideal of $\Bbbk[x_1, \dots, x_{i+1}]$ generated by its monomials of degree m. As a result, $A_0 = K_{i+1}\Bbbk[x_1, \dots, x_{i+1}]$ since A_0 is a graded algebra.

Let $v_1, \dots, v_l$ be linearly free over $\Bbbk$ in K_{i+1} and let $a_1, \dots, a_l$ be in $\Bbbk[x_1, \dots, x_{i+1}]$ such that

$$a_1 v_1 + \cdots a_l v_l = 0.$$

For $j = 1, \dots, l$, a_j has an expansion

$$a_j = \sum_{m \in \mathbb{N}} a_{j,m} x_{i+1}^m$$

with $a_{j,m}, m \in \mathbb{N}$ in $\Bbbk[x_1, \dots, x_i]$ with finite support. According to Corollary 2.15,(i), the sequence $x_{i+1}^m \mu_i(v_j), (j, m) \in \{1, \dots, l\} \times \mathbb{N}$ is linearly free over $\Bbbk$. So, by induction hypothesis, $a_{j,m} = 0$ for all (j, m). As a result, A_0 is a free extension of $\Bbbk[x_1, \dots, x_{i+1}]$, whence the claim.

(ii) According to Lemma 2.13,(iii) and (iv), $B_0 = \Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]$ and $B_{k\ell} = \Bbbk[\mathcal{N}^{(k)}]$. So the assertion will result from the following claim:

Claim 2.20. For $i = 1, \dots, k\ell$, B_0 is a free extension of $\Bbbk[x_1, \dots, x_i]$ and B_i is the quotient of B_0 by the ideal generated by $x_1, \dots, x_i$.

Prove the claim by induction on i. According to Corollary 2.15,(iii), the claim is true for $i = 1$. Suppose that it is true for i. By Corollary 2.15,(iii), B_{i+1} is the quotient of B_i by the ideal generated by x_{i+1}. So by induction hypothesis, B_{i+1} is the quotient of B_0 by the ideal generated by $x_1, \dots, x_{i+1}$. For $j = 0, \dots, k\ell$, B_j is contained in A_j and the quotient morphism $B_0 \longrightarrow B_j$ is the restriction of μ_j to B_0. Let K'_{i+1} be a $\Bbbk$-subspace of B_0 such that the restriction of μ_{i+1} to K'_{i+1} is a $\Bbbk$-linear isomorphism onto B_{i+1}. Then $B_0 = K'_{i+1} + B_0 x_1 + \cdots + B_0 x_{i+1}$. So by induction on m,

$$B_0 = K'_{i+1}\Bbbk[x_1, \dots, x_{i+1}] + B_0 \mathfrak{I}_m.$$

As a result, $B_0 = K'_{i+1}\Bbbk[x_1, \dots, x_{i+1}]$ since B_0 is a graded algebra. Moreover by (i), a basis of K'_{i+1} is linearly free over $\Bbbk[x_1, \dots, x_{i+1}]$ so that B_0 is a free extension of $\Bbbk[x_1, \dots, x_{i+1}]$, whence the claim. □

Remark 2.21. According to Proposition 2.18, $\mathrm{S}(\mathfrak{h}^k)$ is embedded in $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]$ and by Lemma 2.13,(iii), the embedding is given by the map

$$\mathrm{S}(\mathfrak{h}^k) \longrightarrow \Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}] , \qquad p \longmapsto ((x_1,\ldots,x_k,y_1,\ldots,y_k) \mapsto p(y_1,\ldots,y_k).$$

Denote by Φ this map.

Corollary 2.22. (i) *The image of Φ is equal to $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]^G$. Moreover, $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]$ is generated by $\Bbbk[\mathcal{B}^{(k)}]$ and $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]^G$.*

(ii) *The image of Φ is equal to $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]^G$.*

(iii) *The subalgebras $\Bbbk[\mathcal{B}^{(k)}]^G$ and $\Phi((\mathrm{S}(\mathfrak{h})^{\otimes k})^{W(\mathcal{R})})$ of $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]^G$ are equal.*

Proof. (i) Since $\mathcal{B}_{\mathrm{x}}^{(k)}$ is a closed subvariety of $\mathcal{X}^k$ and $\Bbbk[\mathcal{X}]$ is generated by $\mathrm{S}(\mathfrak{g})$ and $\mathrm{S}(\mathfrak{h})$, $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]$ is generated by $\mathrm{S}(\mathfrak{h}^k)$ and the image of $\mathrm{S}(\mathfrak{g}^k)$ in $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]$ which is equal to $\Bbbk[\mathcal{B}^{(k)}]$. For p in $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]$, denote by $\overline{p}$ the element of $\mathrm{S}(\mathfrak{h})^{\otimes k}$ such that

$$\overline{p}(x_1,\ldots,x_k) := p(x_1,\ldots,x_k,x_1,\ldots,x_k).$$

Then the restriction of $p-\Phi(\overline{p})$ to $\iota_k(\mathfrak{h}^k)$ is equal to 0. Moreover, if p is in $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]^G$, $p-\Phi(\overline{p})$ is G-invariant so that $p=\Phi(\overline{p})$ since $G.\iota_k(\mathfrak{h}^k)$ is dense in $\mathcal{B}_{\mathrm{x}}^{(k)}$, whence the assertion.

(ii) Since $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]$ is contained in $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$, so is $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]^G$ by (i). Since G is reductive, there exists a projection $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}] \longrightarrow \Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]^G$ which is $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]^G$-linear. As a result, $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]^G$ is the integral extension of $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]^G$ in $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ since $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$ is an integral extension of $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]$. Let J be the ideal of augmentation of $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]^G$ and set $J' := \Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]J$. By (i) and Proposition 2.18,(i), J' is a prime ideal. Suppose that $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]^G$ is strictly contained in $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]^G$. A contradiction is expected. The algebras $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]^G$ and $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]^G$ are graded subalgebras of $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]$. Let a be a homogeneous element in $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]^G \setminus \Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]^G$ of minimal degree. Then a has positive degree. As a result, it is in J' since J' is radical and a satifies a dependence integral equation over $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]^G$. Since $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]^G J$ is the image of J' by the projection $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}] \longrightarrow \Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]^G$, a is in $\Bbbk[\mathcal{B}_{\mathrm{n}}^{(k)}]^G J$. By the minimality of the degree of a, a is in $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]^G$, whence the contradiction.

(iii) For $(x_1,\ldots,x_k)$ in $\mathfrak{h}^k$, for w in $W(\mathcal{R})$ and for g_w a representative of w in $N_G(\mathfrak{h})$, we have

$$(w(x_1),\ldots,w(x_k),w(x_1),\ldots,w(x_k)) = g_w.(x_1,\ldots,x_k,w(x_1),\ldots,w(x_k))$$

so that the subalgebra $\Bbbk[\mathcal{B}^{(k)}]^G$ of $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]^G$ is contained in $\Phi((\mathrm{S}(\mathfrak{h}^k))^{W(\mathcal{R})})$ by (i). Moreover, since G is reductive, $\Bbbk[\mathcal{B}^{(k)}]^G$ is the image of $(\mathrm{S}(\mathfrak{g})^{\otimes k})^G$ by the restriction morphism. According to [J07, Theorem 2.9 and some remark], the restriction morphism $(\mathrm{S}(\mathfrak{g})^{\otimes k})^G \to (\mathrm{S}(\mathfrak{h})^{\otimes k})^{W(\mathcal{R})}$ is surjective, whence the equality $\Bbbk[\mathcal{B}^{(k)}]^G = \Phi((\mathrm{S}(\mathfrak{h}^k))^{W(\mathcal{R})})$. □

According to Proposition 2.18,(ii) and Corollary 2.22,(i), $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]$ is a free extension of $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]^G = \mathrm{S}(\mathfrak{h}^k)$.

Corollary 2.23. *Let M be a graded complement to $\Bbbk[\mathcal{B}^{(k)}]_+^G\Bbbk[\mathcal{B}^{(k)}]$ in $\Bbbk[\mathcal{B}^{(k)}]$.*

(i) *The space M contains a basis of $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]$ over $\mathrm{S}(\mathfrak{h})^{\otimes k}$.*

(ii) *The intersection of M and $\mathrm{S}_+(\mathfrak{h}^k)\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]$ is different from $\{0\}$.*

Proof. (i) Since M is a graded complement to $\Bbbk[\mathcal{B}^{(k)}]_+^G\Bbbk[\mathcal{B}^{(k)}]$ in $\Bbbk[\mathcal{B}^{(k)}]$, by induction on l,

$$\Bbbk[\mathcal{B}^{(k)}] = M\Bbbk[\mathcal{B}^{(k)}]^G + (\Bbbk[\mathcal{B}^{(k)}]_+^G)^l\Bbbk[\mathcal{B}^{(k)}].$$

Hence $\Bbbk[\mathcal{B}^{(k)}] = M\Bbbk[\mathcal{B}^{(k)}]^G$ since $\Bbbk[\mathcal{B}^{(k)}]$ is graded. Then, by Corollary 2.22,(i) and (iii),

$$\Bbbk[\mathcal{B}_{\rm x}^{(k)}] = M\mathrm{S}(\mathfrak{h})^{\otimes k} \quad \text{and} \quad \Bbbk[\mathcal{B}_{\rm x}^{(k)}] = M + \mathrm{S}_+(\mathfrak{h}^k)\Bbbk[\mathcal{B}_{\rm x}^{(k)}].$$

Then M contains a graded complement M' to $\mathrm{S}_+(\mathfrak{h}^k)\Bbbk[\mathcal{B}_{\rm x}^{(k)}]$ in $\Bbbk[\mathcal{B}_{\rm x}^{(k)}]$, whence the assertion.

(ii) Suppose that $M' = M$. We expect a contradiction. According to (i), the canonical maps

$$M \otimes_{\Bbbk} \mathrm{S}(\mathfrak{h})^{\otimes k} \longrightarrow \Bbbk[\mathcal{B}_{\rm x}^{(k)}] , \qquad M \otimes_{\Bbbk} \Bbbk[\mathcal{B}^{(k)}]^G \longrightarrow \Bbbk[\mathcal{B}^{(k)}]$$

are isomorphisms. Then, according to Lemma 1.5, there exists a group action of $W(\mathcal{R})$ on $\Bbbk[\mathcal{B}_{\rm x}^{(k)}]$ extending the diagonal action of $W(\mathcal{R})$ in $\mathrm{S}(\mathfrak{h})^{\otimes k}$ and such that $\Bbbk[\mathcal{B}_{\rm x}^{(k)}]^{W(\mathcal{R})} = \Bbbk[\mathcal{B}^{(k)}]$ since $\Bbbk[\mathcal{B}^{(k)}] \cap \mathrm{S}(\mathfrak{h})^{\otimes k} = (\mathrm{S}(\mathfrak{h})^{\otimes k})^{W(\mathcal{R})}$ by Corollary 2.22,(iii). Moreover, since $W(\mathcal{R})$ is finite, the subfield of invariant elements of the fraction field of $\Bbbk[\mathcal{B}_{\rm x}^{(k)}]$ is the fraction field of $\Bbbk[\mathcal{B}_{\rm x}^{(k)}]^{W(\mathcal{R})}$. Hence the action of $W(\mathcal{R})$ in $\Bbbk[\mathcal{B}_{\rm x}^{(k)}]$ is trivial since $\Bbbk[\mathcal{B}_{\rm x}^{(k)}]$ and $\Bbbk[\mathcal{B}^{(k)}]$ have the same fraction field, whence the contradiction since $(\mathrm{S}(\mathfrak{h})^{\otimes k})^{W(\mathcal{R})}$ is strictly contained in $\mathrm{S}(\mathfrak{h})^{\otimes k}$. □

3. On the nullcone

Let $k \geq 2$ be an integer. Let I be the ideal of $\Bbbk[\mathcal{B}_{\rm n}^{(k)}]$ generated by $1 \otimes \mathrm{S}_+(\mathfrak{h}^k)$.

Lemma 3.1. *Let N be the subscheme of $\mathcal{B}_{\rm n}^{(k)}$ defined by I.*

(i) *The ideal I is prime and N is isomorphic to $\mathcal{N}_{\rm n}^{(k)}$.*

(ii) *The variety N is the inverse image of $\mathcal{N}^{(k)}$ by $\eta_{\rm n}$.*

Proof. (i) By Proposition 2.18,(i), $\Bbbk[N] = \Bbbk[\mathcal{N}_{\rm n}^{(k)}]$, whence the assertion.

(ii) By (i), N is reduced hence a variety. According to Remark 2.21, for $(g, x_1, \ldots, x_k)$ in $G \times \mathfrak{b}^k$, $\gamma_{\rm n}(\overline{(g, x_1, \ldots, x_k)})$ is a zero of I if and only if $x_1, \ldots, x_k$ are nilpotent, whence the assertion. □

Theorem 3.2. (i) *The variety $\mathcal{N}^{(k)}$ is normal if and only if so is $\mathcal{B}_{\rm x}^{(k)}$. If so, $\gamma_{\rm n} = \gamma_{\rm x}$ and the restriction of ϖ to $\mathcal{B}_{\rm x}^{(k)}$ is the normalization morphism of $\mathcal{B}^{(k)}$.*

(ii) *The variety $\mathcal{N}^{(k)}$ is Cohen-Macaulay if and only if so is $\mathcal{B}_{\rm x}^{(k)}$.*

(iii) *The variety $\mathcal{N}^{(k)}$ has rational singularities if and only if it is Cohen-Macaulay.*

(iv) *The variety $\mathcal{B}_{\rm x}^{(k)}$ has rational singularities if and only if it is Cohen-Macaulay.*

(v) *Let I_0 be the ideal of $\Bbbk[\mathcal{B}^{(k)}]$ generated by $\Bbbk[\mathcal{B}^{(k)}]_+^G$. Then I_0 is strictly contained in the ideal of definition of $\mathcal{N}^{(k)}$ in $\Bbbk[\mathcal{B}^{(k)}]$.*

Proof. (i) According to Proposition 2.18,(ii), $\Bbbk[\mathcal{B}_{\rm x}^{(k)}]$ is a free extension of $\mathrm{S}(\mathfrak{h}^k)$ and $\Bbbk[\mathcal{N}^{(k)}]$ is the quotient of $\Bbbk[\mathcal{B}_{\rm x}^{(k)}]$ by the ideal generated by $\mathrm{S}_+(\mathfrak{h}^k)$. So by [MA86, Ch. 8, Theorem 23.9 and Corollary], 0 is a normal point of $\mathcal{B}_{\rm x}^{(k)}$ if $\mathcal{N}^{(k)}$ is normal. As a result $\mathcal{B}_{\rm x}^{(k)}$ is normal if so is $\mathcal{N}^{(k)}$ since $\mathcal{B}_{\rm x}^{(k)}$ is a cone and its set of normal points is open. Conversely, suppose that $\mathcal{B}_{\rm x}^{(k)}$ is normal so that $\mathcal{B}_{\rm n}^{(k)} = \mathcal{B}_{\rm x}^{(k)}$ and $\gamma_{\rm n} = \gamma_{\rm x}$. Moreover, by Corollary 2.8,(i), the restriction of ϖ to $\mathcal{B}_{\rm x}^{(k)}$ is the normalization morphism of $\mathcal{B}^{(k)}$. According to Proposition 2.18,(i), $\Bbbk[\mathcal{N}_{\rm n}^{(k)}]$ is the image of $\Bbbk[\mathcal{B}_{\rm n}^{(k)}]$ by a morphism and by Proposition 2.18,(ii), $\Bbbk[\mathcal{N}^{(k)}]$ is the image of $\Bbbk[\mathcal{B}_{\rm x}^{(k)}]$ by this morphism, whence $\Bbbk[\mathcal{N}^{(k)}] = \Bbbk[\mathcal{N}_{\rm n}^{(k)}]$.

(ii) Suppose that $\mathcal{N}^{(k)}$ is Cohen-Macaulay. Then the localization of $\Bbbk[\mathcal{B}_{\rm x}^{(k)}]$ at 0 is Cohen-Macaulay by [MA86, Ch. 8, Theorem 23.9] and Proposition 2.18,(ii). By [MA86, Ch. 8, Theorem 24.5], the set of points of $\mathcal{B}_{\rm x}^{(k)}$ at which the localization is Cohen-Macaulay is open. Hence $\mathcal{B}_{\rm x}^{(k)}$ is Cohen-Macaulay since its is a cone.

Conversely suppose that $\mathcal{B}_{\rm x}^{(k)}$ is Cohen-Macaulay. According to Proposition 2.18,(ii), any basis in $\mathrm{S}(\mathfrak{h}^k)$ is a regular sequence in $\Bbbk[\mathcal{B}_{\rm x}^{(k)}]$ and $\Bbbk[\mathcal{N}^{(k)}]$ is the quotient of $\Bbbk[\mathcal{B}_{\rm x}^{(k)}]$ by the ideal generated by this sequence. Then the localization at 0 of $\Bbbk[\mathcal{N}^{(k)}]$ is Cohen-Macaulay by [MA86, Ch .6, Theorem 17.4 and Ch. 5, Theorem 14.1]. So, again by [MA86, Ch. 8, Theorem 24.5], $\mathcal{N}^{(k)}$ is Cohen-Macaulay since its is a cone.

(iii) According to [KK73, p. 50], $\mathcal{N}^{(k)}$ is Cohen-Macaulay if it has rational singularities. Suppose that $\mathcal{N}^{(k)}$ is Cohen-Macaulay. By Lemma 2.2,(iv) and Corollary 2.3,(i), $\mathcal{N}^{(k)}$ is smooth in codimension 1. Then, by Serre's normality criterion [Bou98, §1,no 10, Théorème 4], $\mathcal{N}^{(k)}$ is normal. So, by [KK73, p.50], it remains to prove that for U open subset of $\mathcal{N}^{(k)}$ and ω a regular differential form of top degree on the smooth locus of U, $\upsilon^*(\omega)$ has a regular extension to $\upsilon^{-1}(U)$.

Let U' be the smooth locus of U. According to Lemma 2.2,(iv), $U \cap V_k$ is contained in U'. So by Corollary 2.3,(iii), $\upsilon^{-1}(U')$ is a big open subset of $\upsilon^{-1}(U)$. Let $\Omega_{\upsilon^{-1}(U)}$ be the sheaf of regular differential forms of top degree on $\upsilon^{-1}(U)$. For some open cover $O_1, \ldots, O_m$ of $\upsilon^{-1}(U)$, for $i = 1, \ldots, m$, the restriction of $\Omega_{\upsilon^{-1}(U)}$ to O_i is a free $\mathcal{O}_{O_i}$-module of rank 1. Denoting by ω_i a generator, for some regular function a_i on $O_i \cap \upsilon^{-1}(U')$,

$$\omega\,|_{O_i \cap \upsilon^{-1}(U')} = a_i(\omega_i\,|_{O_i \cap \upsilon^{-1}(U')}).$$

Since O_i is normal and $O_i \cap \upsilon^{-1}(U')$ is a big open subset of O_i, a_i has a regular extension to O_i. Denoting again by a_i this extension, $a_i\omega_i$ is a regular differential form of top degree on O_i having the same restriction as $\upsilon^*(\omega)$ to $O_i \cap \upsilon^{-1}(U')$. As a result, since $\Omega_{\upsilon^{-1}(U)}$ is torsion free and $\upsilon^{-1}(U)$ is irreducible, for $1 \leq i, j \leq m$, $a_i\omega_i$ and $a_j\omega_j$ have the same restriction to $O_i \cap O_j$. Denoting by ω' the global section of $\Omega_{\upsilon^{-1}(U)}$ whose restriction to O_i is $a_i\omega_i$ for $i = 1, \ldots, m$, $\upsilon^*(\omega)$ is the restriction of ω' to $\upsilon^{-1}(U')$, whence the assertion.

(iv) According to [KK73, p. 50], $\mathcal{B}_{\rm x}^{(k)}$ is Cohen-Macaulay if it has rational singularities. Suppose that $\mathcal{B}_{\rm x}^{(k)}$ is Cohen-Macaulay. By Lemma 2.10,(iv), $\mathcal{B}_{\rm x}^{(k)}$ is smooth in codimension 1. Then, by Serre's normality criterion [Bou98, §1,no 10, Théorème 4], $\mathcal{B}_{\rm x}^{(k)}$ is normal. So, by [KK73, p.50], it remains to prove that for U open subset of $\mathcal{B}_{\rm x}^{(k)}$ and ω a regular differential form of top degree on the smooth locus of U, $\gamma_{\rm x}^*(\omega)$ has a regular extension to $\gamma_{\rm x}^{-1}(U)$.

Let U' be the smooth locus of U. According to Lemma 2.10,(iv), $U \cap W_k$ is contained in U'. So by Corollary 2.11, $\gamma_{\rm x}^{-1}(U')$ is a big open subset of $\gamma_{\rm x}^{-1}(U)$. Let $\Omega_{\gamma_{\rm x}^{-1}(U)}$ be the sheaf of regular differential forms of top degree on $\gamma_{\rm x}^{-1}(U)$. For some open cover $O_1, \ldots, O_m$ of $\gamma_{\rm x}^{-1}(U)$, for $i = 1, \ldots, m$, the restriction of $\Omega_{\gamma_{\rm x}^{-1}(U)}$ to O_i is a free $\mathcal{O}_{O_i}$-module of rank 1. Denoting by ω_i a generator, for some regular function a_i on $O_i \cap \gamma_{\rm x}^{-1}(U')$,

$$\omega\,|_{O_i \cap \gamma_{\rm x}^{-1}(U')} = a_i(\omega_i\,|_{O_i \cap \gamma_{\rm x}^{-1}(U')}).$$

Since O_i is normal and $O_i \cap \gamma_{\rm x}^{-1}(U')$ is a big open subset of O_i, a_i has a regular extension to O_i. Denoting again by a_i this extension, $a_i\omega_i$ is a regular differential form of top degree on O_i having the same restriction as $\gamma_{\rm x}^*(\omega)$ to $O_i \cap \gamma_{\rm x}^{-1}(U')$. As a result, since $\Omega_{\gamma_{\rm x}^{-1}(U)}$ is torsion free and $\gamma_{\rm x}^{-1}(U)$ is irreducible, for $1 \leq i, j \leq m$, $a_i\omega_i$ and $a_j\omega_j$ have the same restriction to $O_i \cap O_j$. Denoting by ω' the global section of $\Omega_{\gamma_{\rm x}^{-1}(U)}$ whose restriction to O_i is $a_i\omega_i$ for $i = 1, \ldots, m$, $\gamma_{\rm x}^*(\omega)$ is the restriction of ω' to $\gamma_{\rm x}^{-1}(U')$, whence the assertion.

(v) Since $\Bbbk[\mathcal{B}^{(k)}]_+^G$ is contained in $\mathrm{S}_+(\mathfrak{h}^k)$, I_0 is contained in $I \cap \Bbbk[\mathcal{B}^{(k)}]$. According to Lemma 3.1,(ii) and (i), $I \cap \Bbbk[\mathcal{B}^{(k)}]$ is the ideal of definition of $\mathcal{N}^{(k)}$ in $\Bbbk[\mathcal{B}^{(k)}]$. Let M be a graded complement of $\Bbbk[\mathcal{B}^{(k)}]_+^G \Bbbk[\mathcal{B}^{(k)}]$ in $\Bbbk[\mathcal{B}^{(k)}]$. According to Corollary 2.23,(ii), $I \cap M$ is different from $\{0\}$. Hence I_0 is strictly contained in $I \cap \Bbbk[\mathcal{B}^{(k)}]$, whence the assertion. □

Remark 3.3. According to [VX15, 6.2], for $\mathfrak{g}$ simple of type B_2, $\mathcal{N}^{(2)}$ is not normal and according to [VX15, Theorem 6.1], for $\mathfrak{g} = \mathrm{sl}_3$, $\mathcal{N}^{(k)}$ has rational singularities for all k.

Summarizing the results of the preceding subsections, Theorem 1.1,(i), (ii), (iii), (iv), (vii) are given by Theorem 3.2, Theorem 1.1,(v) is given by Proposition 2.18,(ii) and Corollary 2.22,(i) and Theorem 1.1,(vi) is given by Corollary 2.22,(iii). To complete Theorem 1.1, recall that $\varkappa$ is the normalization morphism of $\mathcal{N}^{(k)}$ and denote by η the normalization morphism of $\mathcal{B}_{\mathrm{x}}^{(k)}$.

Proposition 3.4. (i) *The morphism* η *is a homeomorphism.*
(ii) *The morphism* $\varkappa$ *is a homeomorphism.*

Proof. Recall that the morphisms

$$G\times_B \mathfrak{b} \longrightarrow G/B\times\mathfrak{g} \qquad \text{and} \qquad G\times_B \mathfrak{b}^k \longrightarrow G/B\times\mathfrak{g}^k$$

are closed embeddings. For $x=(x_1,\ldots,x_k,y_1,\ldots,y_k)$ in $\mathcal{B}_{\mathrm{x}}^{(k)}$, denote by $\mathfrak{B}_x$ the subset of Borel subalgebras $\mathfrak{b}'$ of $\mathfrak{g}$ such that $\chi_{\mathrm{n}}(\mathfrak{b}',x_i)=(x_i,y_i)$ for $i=1,\ldots,k$. Then $\gamma_{\mathrm{x}}^{-1}(\{x\})=\mathfrak{B}_x\times\{(x_1,\ldots,x)\}$. From the two commutative diagramms

$$\begin{tikzcd} G\times_B\mathfrak{b}^k \arrow[rr, "\gamma_{\mathrm{n}}"] \arrow[dr, "\gamma_{\mathrm{x}}"'] & & \mathcal{B}_{\mathrm{n}}^{(k)} \arrow[dl, "\eta"] \\ & \mathcal{B}_{\mathrm{x}}^{(k)} & \end{tikzcd}, \qquad \begin{tikzcd} G\times_B\mathfrak{u}^k \arrow[rr, "\upsilon_{\mathrm{n}}"] \arrow[dr, "\upsilon"'] & & \mathcal{N}_{\mathrm{n}}^{(k)} \arrow[dl, "\varkappa"] \\ & \mathcal{N}^{(k)} & \end{tikzcd}$$

we deduce that it suffices to prove that $\mathfrak{B}_x$ is connected for all x in $\mathcal{B}_{\mathrm{x}}^{(k)}$ since υ is the restriction of γ_{x} to $G\times_B\mathfrak{u}^k$.

For β a simple root, denote by s_β the reflection of $\mathfrak{h}$ with respect to β. For w in $W(\mathcal{R})$ denote by $l(w)$ its length with respect to the set of simple roots, n_w a representative of w in $N_G(\mathfrak{h})$ and set $w(\mathfrak{b}):=n_w(\mathfrak{b})$. Let $x=(x_1,\ldots,x_k,y_1,\ldots,y_k)$ be in $\mathcal{B}_{\mathrm{x}}^{(k)}$ and $\mathfrak{b}'$ and $\mathfrak{b}''$ in $\mathfrak{B}_x$. By Bruhat decomposition of G, for some (g,b,w) in $G\times B\times W(\mathcal{R})$, $\mathfrak{b}'=g(\mathfrak{b})$, $\mathfrak{b}''=gbw(\mathfrak{b})$. Set:

$$l(w):=q, \qquad u_i:=g^{-1}(x_i), \qquad v_i:=b^{-1}(u_i)$$

for $i=1,\ldots,k$. In particular, $v:=(v_1,\ldots,v_k,y_1,\ldots,y_k)$ is in $\mathcal{B}_{\mathrm{x}}^{(k)}$ and $\mathfrak{b}$ and $w(\mathfrak{b})$ are in $\mathfrak{B}_v$ and it suffices to prove $\mathfrak{b}$ and $w(\mathfrak{b})$ are in the same connected component of $\mathfrak{B}_v$ since $x=gb.v$. It will be a consequence of the following claim.

Claim 3.5. There exist a sequence $L_1,\ldots,L_q$ of projective lines contained in $\mathfrak{B}_v$ and a sequence $\mathfrak{b}_0,\ldots,\mathfrak{b}_q$ in $\mathfrak{B}_v$ such that

$$\mathfrak{b}=\mathfrak{b}_0, \quad w(\mathfrak{b})=\mathfrak{b}_q, \quad \mathfrak{b}_{i-1}\in L_i, \quad \mathfrak{b}_i\in L_i$$

for $i=1,\ldots,q$.

Prove the claim by induction on q. For $q=0$, $\mathfrak{b}=w(\mathfrak{b})$. Suppose that $q=1$ and $w=s_\beta$ for some simple root β. Then $v_1,\ldots,v_k$ are in $\mathfrak{b}\cap s_\beta(\mathfrak{b})$ and for $i=1,\ldots,k$,

$$\chi_{\mathrm{n}}(\mathfrak{b},v_i)=\chi_{\mathrm{n}}(s_\beta(\mathfrak{b}),v_i)=(v_i,y_i).$$

Let $\mathfrak{p}_\beta$ be the parabolic subalgebra $\mathfrak{g}^{-\beta}+\mathfrak{b}$ and $\mathfrak{l}_\beta$ the reductive factor containing $\mathfrak{h}$. The set L_β of Borel subalgebras of $\mathfrak{g}$, contained in $\mathfrak{p}_\beta$, is a projective line. In the case $\mathfrak{g}=\mathfrak{l}_\beta$, $\mathcal{N}^{(k)}$ is normal and η is an isomorphism by Theorem 3.2,(i). Then, by Zariski's Main Theorem [Mu88, §9], the fibers of γ_{x} are connected. So, L_β is contained in $\mathfrak{B}_v$ since $\mathfrak{b}$ and $s_\beta(\mathfrak{b})$ are two different points of $\mathfrak{B}_v\cap L_\beta$.

Suppose the claim true for the integers smaller than q. Let $w = s_1\cdots s_q$ be a reduced decomposition of w and set $w' := s_1\cdots s_{q-1}$. For $i = 1,\ldots,q$, let β_i be the simple root such that $s_i = s_{\beta_i}$. For $i = 1,\ldots,k$,

$$v_i \in \mathfrak{h} \oplus \bigoplus_{\substack{\gamma\in\mathcal{R}_+ \\ w(\gamma)\in\mathcal{R}_+}} \mathfrak{g}^{w(\gamma)} \quad \text{and} \quad n_{w'}^{-1}(v_i) \in \mathfrak{h} \oplus \bigoplus_{\substack{\gamma\in\mathcal{R}_+ \\ w(\gamma)\in\mathcal{R}_+}} \mathfrak{g}^{s_q(\gamma)}.$$

Since $s_1\cdots s_q$ is the reduced decomposition of w, $w(\beta_q)$ is a negative root, whence

$$\{\gamma \in \mathcal{R}_+ \,|\, w(\gamma) \in \mathcal{R}_+\} \subset \mathcal{R}_+ \setminus \{\beta_q\}.$$

As a result $n_{w'}^{-1}(v_1),\ldots,n_{w'}^{-1}(v_k)$ are in $\mathfrak{b}$. So, by induction hypothesis, there exist a sequence $L_0,\ldots,L_{q-1}$ of projective lines contained in $\mathfrak{B}_v$ and a sequence $\mathfrak{b}_0,\ldots,\mathfrak{b}_{q-1}$ in $\mathfrak{B}_v$ such that

$$\mathfrak{b} = \mathfrak{b}_0, \qquad w'(\mathfrak{b}) = \mathfrak{b}_{q-1}, \qquad \mathfrak{b}_{i-1} \in L_i, \qquad \mathfrak{b}_i \in L_i$$

for $i = 1,\ldots,q-1$. By the case $q = 1$, for some projective line L'_q, contained in $\mathfrak{B}_{n_{w'}^{-1}.v}$, $\mathfrak{b}$ and $s_q(\mathfrak{b})$ are in L'_q. Then, setting $\mathfrak{b}_q = w(\mathfrak{b})$ and $L_q := n_{w'}.L'_q$, the sequences $L_1,\ldots,L_q$ and $\mathfrak{b}_0,\ldots,\mathfrak{b}_q$ verify the conditions of the claim. □

4. Main varieties

Denote by X the closure in $\mathrm{Gr}_\ell(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under B. According to Lemma 1.7, $G.X$ is the closure in $\mathrm{Gr}_\ell(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under G. Let $\mathcal{E}_0$ and $\mathcal{E}$ be the restrictions to X and $G.X$ respectively of the totaulogical vector bundle over $\mathrm{Gr}_\ell(\mathfrak{g})$. By definition, $\mathcal{E}$ is the subvariety of elements (V,x) of $G.X \times \mathfrak{g}$ such that x is in V and $\mathcal{E}_0$ is the intersection of $\mathcal{E}$ and $X \times \mathfrak{b}$. In this section, we give geometric properties of X and $G.X$. These varieties play an important role in the study of the generalized commuting varieties and isospectrale commuting varieties as it is suggested by Theorem 1.3 and it will be shown in two future notes.

4.1. For α in $\mathcal{R}$, set $V_\alpha := \mathfrak{h}_\alpha \oplus \mathfrak{g}^\alpha$ and denote by X_α the closure in $\mathrm{Gr}_\ell(\mathfrak{g})$ of the orbit of V_α under B.

Lemma 4.1. *Let α be in $\mathcal{R}_+$. Let $\mathfrak{p}$ be a parabolic subalgebra containing $\mathfrak{b}$ and let P be its normalizer in G.*

(i) *The subset $P.X$ of $\mathrm{Gr}_\ell(\mathfrak{g})$ is the closure in $\mathrm{Gr}_\ell(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under P.*

(ii) *The closed set X_α of $\mathrm{Gr}_\ell(\mathfrak{g})$ is an irreducible component of $X \setminus B.\mathfrak{h}$.*

(iii) *The set $P.X_\alpha$ is an irreducible component of $P.X \setminus P.\mathfrak{h}$.*

(iv) *The varieties $X \setminus B.\mathfrak{h}$ and $P.X \setminus P.\mathfrak{h}$ are equidimenional of codimension 1 in X and $P.X$ respectively.*

Proof. (i) Since X is a B-invariant closed subset of $\mathrm{Gr}_\ell(\mathfrak{g})$, $P.X$ is a closed subset of $\mathrm{Gr}_\ell(\mathfrak{g})$ by Lemma 1.7. Hence $\overline{P.\mathfrak{h}}$ is contained in $P.X$ since $\mathfrak{h}$ is in X, whence the assertion since $\overline{P.\mathfrak{h}}$ is a P-invariant subset containing X.

(ii) Denoting by H_α the coroot of α,

$$\lim_{t\to\infty} \exp(t\,\mathrm{ad}\, x_\alpha)(\frac{-1}{2t}H_\alpha) = x_\alpha.$$

So V_α is in the closure of the orbit of $\mathfrak{h}$ under the one parameter subgroup of G generated by $\mathrm{ad}\, x_\alpha$. As a result, X_α is a closed subset of $X \setminus B.\mathfrak{h}$ since V_α is not a Cartan subalgebra. Moreover, X_α has dimension $n-1$ since the normalizer of V_α in $\mathfrak{g}$ is $\mathfrak{h} + \mathfrak{g}^\alpha$. Hence X_α is an irreducible component of $X \setminus B.\mathfrak{h}$ since X has dimension n.

(iii) Since X_α is a B-invariant closed subset of $\mathrm{Gr}_\ell(\mathfrak{g})$, $P.X_\alpha$ is a closed subset of $\mathrm{Gr}_\ell(\mathfrak{g})$ by Lemma 1.7. According to (ii), $P.X_\alpha$ is contained in $P.X \setminus P.\mathfrak{h}$ and it has dimension $\dim \mathfrak{p} - \ell - 1$, whence the assertion since $P.X$ has dimension $\dim \mathfrak{p} - \ell$.

(iv) Let P_{u} be the unipotent radical of P and let L be the reductive factor of P whose Lie algebra contains $\mathrm{ad}\,\mathfrak{h}$. Denote by $N_L(\mathfrak{h})$ the normalizer of $\mathfrak{h}$ in L. Since $B.\mathfrak{h}$ and $P.\mathfrak{h}$ are isomorphic to U and $L/N_L(\mathfrak{h}) \times P_{\mathrm{u}}$ respectively, they are affine open subsets of X and $P.X$ respectively, whence the assertion by [EGAIV, Corollaire 21.12.7]. □

For x in $\mathfrak{g}$, set:

$$V_x := \mathrm{span}(\{\varepsilon_1(x), \dots, \varepsilon_\ell(x)\}).$$

Lemma 4.2. (i) *For (V, x) in $X \times \mathfrak{b}$, (V, x) is in the closure of $B.(\{\mathfrak{h}\} \times \mathfrak{h}_{\mathrm{reg}})$ in $\mathrm{Gr}_\ell(\mathfrak{b}) \times \mathfrak{b}$ if and only if x is in V.*

(ii) *The set $\mathcal{E}$ is the closure in $\mathrm{Gr}_\ell(\mathfrak{g}) \times \mathfrak{g}$ of $G.(\{\mathfrak{h}\} \times \mathfrak{h}_{\mathrm{reg}})$.*

(iii) *For (V, x) in $\mathcal{E}$, V_x is contained in V.*

Proof. (i) Let $\mathcal{E}'_0$ be the closure of $B.(\{\mathfrak{h}\} \times \mathfrak{h}_{\mathrm{reg}})$ in $\mathrm{Gr}_\ell(\mathfrak{b}) \times \mathfrak{b}$. Then $\mathcal{E}'_0$ is a closed subset of $\mathcal{E}_0$. Let (V, x) be in $\mathcal{E}_0$. Let E be a complement to V in $\mathfrak{b}$ and let Ω_E be the set of complements to E in $\mathfrak{g}$. Then Ω_E is an open neighborhood of V in $\mathrm{Gr}_\ell(\mathfrak{b})$. Moreover, the map

$$\mathrm{Hom}_{\Bbbk}(V, E) \xrightarrow{\kappa} \Omega_E \ , \qquad \varphi \longmapsto \kappa(\varphi) := \mathrm{span}(\{v + \varphi(v) \mid v \in V\}).$$

is an isomorphism of varieties. Let Ω_E^c be the inverse image of the set of Cartan subalgebras. Then 0 is in the closure of Ω_E^c in $\mathrm{Hom}_{\Bbbk}(V, E)$ since V is in X. For all φ in Ω_E^c, $(\kappa(\varphi), x + \varphi(x))$ is in $\mathcal{E}'_0$. Hence (V, x) is in $\mathcal{E}'_0$.

(ii) Let (V, x) be in $\mathcal{E}$. For some g in G, $g(V)$ is in X. So by (i), $(g(V), g(x))$ is in $\mathcal{E}_0$ and (V, x) is in the closure of $G.(\{\mathfrak{h}\} \times \mathfrak{h}_{\mathrm{reg}})$ in $\mathrm{Gr}_\ell(\mathfrak{g}) \times \mathfrak{g}$, whence the assertion.

(iii) For $i = 1, \dots, \ell$, let $\mathcal{E}_i$ be the set of elements (V, x) of $\mathcal{E}$ such that $\varepsilon_i(x)$ is in V. Then $\mathcal{E}_i$ is a closed subset of $G.X \times \mathfrak{g}$, invariant under the action of G in $\mathrm{Gr}_\ell(\mathfrak{g}) \times \mathfrak{g}$ since ε_i is a G-equivariant map. For all (g, x) in $G \times \mathfrak{h}_{\mathrm{reg}}$, $(g(\mathfrak{h}), g(x))$ is in $\mathcal{E}_i$ since $\varepsilon_i(g(x))$ centralizes $g(x)$. Hence $\mathcal{E}_i = \mathcal{E}$ since $G.(\mathfrak{h}_{\mathrm{reg}} \times \{\mathfrak{h}\})$ is dense in $\mathcal{E}$ by (ii). As a result, for all V in $G.X$ and for all x in V, $\varepsilon_1(x), \dots, \varepsilon_\ell(x)$ are in V. □

Corollary 4.3. *Let (V, x) be in $\mathcal{E}$.*

(i) *The space $\mathfrak{z}_{x_{\mathrm{s}}}$ is contained in V_x and V.*

(ii) *The space V is an algebraic, commutative subalgebra of $\mathfrak{g}$.*

Proof. (i) If x is regular semisimple, V is a Cartan subalgebra of $\mathfrak{g}$ whence the assertion in this case by Lemma 4.2,(iii) and [Ko63, Theorem 9]. Suppose that x is not regular semisimple. Let $\mathfrak{N}_{\mathfrak{g}^{x_{\mathrm{s}}}}$ be the nilpotent cone of $\mathfrak{g}^{x_s}$ and let Ω_{reg} be the regular nilpotent orbit of $\mathfrak{g}^{x_{\mathrm{s}}}$. For all y in Ω_{reg}, $x_{\mathrm{s}} + y$ is in $\mathfrak{g}_{\mathrm{reg}}$ and $\varepsilon_1(x_{\mathrm{s}} + y), \dots, \varepsilon_\ell(x_{\mathrm{s}} + y)$ is a basis of $\mathfrak{g}^{x_{\mathrm{s}}+y}$ by [Ko63, Theorem 9]. Then for all z in $\mathfrak{z}_{x_{\mathrm{s}}}$, there exist regular functions on Ω_{reg}, $a_{1,z}, \dots, a_{\ell,z}$, such that

$$z = a_{1,z}(y)\varepsilon_1(x_{\mathrm{s}} + y) + \cdots + a_{\ell,z}(y)\varepsilon_\ell(x_{\mathrm{s}} + y)$$

for all y in Ω_{reg}. Furthermore, these functions are uniquely defined by this equality. Since $\mathfrak{N}_{\mathfrak{g}^{x_{\mathrm{s}}}}$ is a normal variety and $\mathfrak{N}_{\mathfrak{g}^{x_{\mathrm{s}}}} \setminus \Omega_{\mathrm{reg}}$ has codimension 2 in $\mathfrak{N}_{\mathfrak{g}^{x_{\mathrm{s}}}}$, the functions $a_{1,z}, \dots, a_{\ell,z}$ have regular extensions to $\mathfrak{N}_{\mathfrak{g}^{x_{\mathrm{s}}}}$. Denoting again by $a_{i,z}$ the regular extension of $a_{i,z}$ for $i = 1, \dots, \ell$,

$$z = a_{1,z}(y)\varepsilon_1(x_{\mathrm{s}} + y) + \cdots + a_{\ell,z}(y)\varepsilon_\ell(x_{\mathrm{s}} + y)$$

for all y in $\mathfrak{N}_{\mathfrak{g}^{x_{\mathrm{s}}}}$. As a result, $\mathfrak{z}_{x_{\mathrm{s}}}$ is contained in V_x. Hence $\mathfrak{z}_{x_{\mathrm{s}}}$ is contained in V by Lemma 4.2,(iii).

(ii) Since the set of commutative subalgebras of dimension ℓ is closed in $\mathrm{Gr}_\ell(\mathfrak{g})$, V is a commutative subalgebra of $\mathfrak{g}$. According to (i), the semisimple and nilpotent components of the elements of V are contained in V. For x in $V \setminus \mathfrak{N}_{\mathfrak{g}}$, all the replica of $x_{\rm s}$ are contained in the center of $\mathfrak{g}^{x_{\rm s}}$. Hence V is an algebraic subalgebra of $\mathfrak{g}$ by (i). □

4.2. For s in $\mathfrak{h}$, denote by X^s the subset of elements of X, contained in $\mathfrak{g}^s$.

Lemma 4.4. *Let s be in $\mathfrak{h}$.*

(i) *The set X^s is the closure in $\mathrm{Gr}_\ell(\mathfrak{g}^s)$ of the orbit of $\mathfrak{h}$ under B^s.*

(ii) *The set of elements of $G.X$ containing $\mathfrak{z}_s$ is the closure in $\mathrm{Gr}_\ell(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under G^s.*

Proof. (i) Set $\mathfrak{p} := \mathfrak{g}^s + \mathfrak{b}$, let P be the normalizer of $\mathfrak{p}$ in G and let $\mathfrak{p}_{\rm u}$ be the nilpotent radical of $\mathfrak{p}$. For g in P, denote by $\overline{g}$ its image by the canonical projection from P to G^s. Let Z be the closure in $\mathrm{Gr}_\ell(\mathfrak{g}) \times \mathrm{Gr}_\ell(\mathfrak{g})$ of the image of the map

$$B \longrightarrow \mathrm{Gr}_\ell(\mathfrak{b}) \times \mathrm{Gr}_\ell(\mathfrak{b}) , \qquad g \longmapsto (g(\mathfrak{h}), \overline{g}(\mathfrak{h}))$$

and let Z' be the subset of elements (V, V') of $\mathrm{Gr}_\ell(\mathfrak{b}) \times \mathrm{Gr}_\ell(\mathfrak{b})$ such that

$$V' \subset \mathfrak{g}^{\rm s} \cap \mathfrak{b} \quad \text{and} \quad V \subset V' \oplus \mathfrak{p}_{\rm u}.$$

Then Z' is a closed subset of $\mathrm{Gr}_\ell(\mathfrak{b}) \times \mathrm{Gr}_\ell(\mathfrak{b})$ and Z is contained in Z' since $(g(\mathfrak{h}), \overline{g}(\mathfrak{h}))$ is in Z' for all g in B. Since $\mathrm{Gr}_\ell(\mathfrak{b})$ is a projective variety, the images of Z by the projections $(V, V') \mapsto V$ and $(V, V') \mapsto V'$ are closed in $\mathrm{Gr}_\ell(\mathfrak{b})$ and they are equal to X and $\overline{B^s.\mathfrak{h}}$ respectively. Furthermore, $\overline{B^s.\mathfrak{h}}$ is contained in X^s.

Let V be in X^s. For some V' in $\mathrm{Gr}_\ell(\mathfrak{b})$, (V, V') is in Z. Since

$$V \subset \mathfrak{g}^s, \ V' \subset \mathfrak{g}^s, \ V \subset V' \oplus \mathfrak{p}_{\rm u},$$

$V = V'$ so that V is in $\overline{B^s.\mathfrak{h}}$, whence the assertion.

(ii) Since $\mathfrak{z}_{\rm s}$ is contained in $\mathfrak{h}$, all element of $\overline{G^s.\mathfrak{h}}$ is an element of $G.X$ containing $\mathfrak{z}_s$. Let V be in $G.X$, containing $\mathfrak{z}_s$. Since V is a commutative subalgebra of $\mathfrak{g}^s$ and since $\mathfrak{g}^s \cap \mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}^s$, for some g in G^s, $g(V)$ is contained in $\mathfrak{b} \cap \mathfrak{g}^s$. So, one can suppose that V is contained in $\mathfrak{b}$. According to the Bruhat decomposition of G, since X is B-invariant, for some b in U and for some w in $W(\mathcal{R})$, V is in $bw.X$. Set:

$$\mathcal{R}_{+,w} := \{\alpha \in \mathcal{R}_+ \mid w(\alpha) \in \mathcal{R}_+\}, \qquad \mathcal{R}'_{+,w} := \{\alpha \in \mathcal{R}_+ \mid w(\alpha) \notin \mathcal{R}_+\},$$

$$\mathfrak{u}_1 := \bigoplus_{\alpha \in \mathcal{R}_{+,w}} \mathfrak{g}^{w(\alpha)}, \qquad \mathfrak{u}_2 := \bigoplus_{\alpha \in -\mathcal{R}'_{+,w}} \mathfrak{g}^{w(\alpha)}, \qquad \mathfrak{u}_3 := \bigoplus_{\alpha \in \mathcal{R}'_{+,w}} \mathfrak{g}^{w(\alpha)},$$

$$B^w := wBw^{-1}, \qquad \mathfrak{b}^w := \mathfrak{h} \oplus \mathfrak{u}_1 \oplus \mathfrak{u}_3,$$

so that $\mathrm{ad}\,\mathfrak{b}^w$ is the Lie algebra of B^w and $w.X$ is the closure in $\mathrm{Gr}_\ell(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under B^w. Moreover, $\mathfrak{u}$ is the direct sum of $\mathfrak{u}_1$ and $\mathfrak{u}_2$. For $i = 1, 2$, denote by U_i the closed subgroup of U whose Lie algebra is $\mathrm{ad}\,\mathfrak{u}_i$. Then $U = U_2 U_1$ and $b = b_2 b_1$ with b_i in U_i for $i = 1, 2$. Since $w^{-1}(\mathfrak{u}_1)$ is contained in $\mathfrak{u}$ and X is invariant under B, $b_2 b_1 w.X = b_2 w.X$. Then $b_2^{-1}(V)$ is in $w.X$ and

$$b_2^{-1}(V) \subset \mathfrak{b} \cap \mathfrak{b}^w = \mathfrak{h} \oplus \mathfrak{u}_1$$

since V is contained in $\mathfrak{b}$. Set:

$$\mathfrak{u}_{2,1} := \mathfrak{u}_2 \cap \mathfrak{g}^s, \qquad \mathfrak{u}_{2,2} := \mathfrak{u}_2 \cap \mathfrak{p}_{\rm u}$$

and for $i = 1, 2$, denote by $U_{2,i}$ the closed subgroup of U_2 whose Lie algebra is $\mathrm{ad}\,\mathfrak{u}_{2,i}$. Then $\mathfrak{u}_2$ is the direct sum of $\mathfrak{u}_{2,1}$ and $\mathfrak{u}_{2,2}$ and $U_2 = U_{2,1} U_{2,2}$ so that $b_2 = b_{2,1} b_{2,2}$ with $b_{2,i}$ in $U_{2,i}$ for $i = 1, 2$.

As a result, $\mathfrak{z}_s$ is contained in $b_{2,1}^{-1}(V)$ and $b_{2,2}^{-1}(\mathfrak{z}_s)$ is contained $\mathfrak{h} \oplus \mathfrak{u}_1$. Hence $b_{2,2}^{-1}(\mathfrak{z}_s) = \mathfrak{z}_s$ since $\mathfrak{u}_1 \cap \mathfrak{u}_{2,2} = \{0\}$.

Suppose $b_{2,2} \neq 1_{\mathfrak{g}}$. We expect a contradiction. For some x in $\mathfrak{u}_{2,2}$, $b_{2,2} = \exp(\mathrm{ad}\, x)$. The space $\mathfrak{u}_{2,2}$ is a direct sum of root spaces since so are $\mathfrak{u}_2$ and $\mathfrak{p}_{\mathrm{u}}$. Let $\alpha_1, \ldots, \alpha_m$ be the positive roots such that the corresponding root spaces are contained in $\mathfrak{u}_{2,2}$. They are ordered so that for $i \leq j$, $\alpha_j - \alpha_i$ is a positive root if it is a root. For $i = 1, \ldots, m$, let c_i be the coordinate of x at x_{α_i} and let i_0 be the smallest integer such that $c_{i_0} \neq 0$. For all z in $\mathfrak{z}_s$,

$$b_{2,2}^{-1}(z) - z - c_{i_0}\alpha_{i_0}(z)x_{\alpha_{i_0}} \in \bigoplus_{j > i_0} \mathfrak{g}^{\alpha_j},$$

whence the contradiction since for some z in $\mathfrak{z}_s$, $\alpha_{i_0}(z) \neq 0$. As a result, $b_{2,1}^{-1}(V)$ is an element of $w.X = \overline{B^w.\mathfrak{h}}$, contained in $\mathfrak{g}^s$. So, by (i), $b_{2,1}^{-1}(V)$ and V are in $\overline{G^s.\mathfrak{h}}$, whence the assertion. □

Define a torus of $\mathfrak{g}$ as a commutative algebraic subalgebra of $\mathfrak{g}$ whose all elements are semisimple. For Λ subset of $\mathcal{R}$, denote by $\mathfrak{h}_\Lambda$ the intersection of the kernels of the elements of Λ.

Corollary 4.5. *Let V be in X. Then for some subset Λ of $\mathcal{R}$ and for some g in B, $g(V)$ is the direct sum of $\mathfrak{h}_\Lambda$ and $g(V) \cap \mathfrak{u}$.*

Proof. By Corollary 4.3,(ii), V is the direct sum of a subtorus of $\mathfrak{b}$ and its intersection with $\mathfrak{u}$. So for some g in B,

$$g(V) = g(V) \cap \mathfrak{h} \oplus g(V) \cap \mathfrak{u}.$$

Let Λ be the set of roots such that $g(V) \cap \mathfrak{h}$ is contained in $\mathfrak{h}_\Lambda$. If $\Lambda = \mathcal{R}$, $g(V)$ is contained in $\mathfrak{u}$. Suppose Λ strictly containd in $\mathcal{R}$. For some s in $g(V) \cap \mathfrak{h}$, $\alpha(s) \neq 0$ for all α in $\mathcal{R} \setminus \Lambda$. Since $g(V)$ is a commutative algebra, $g(V)$ is contained in $\mathfrak{g}^s$. So, by Lemma 4.4,(i), $g(V)$ is in $\overline{B^s.\mathfrak{h}}$. In particular, by Corollary 4.3,(i), $\mathfrak{h}_\Lambda$ is contained in $g(V)$ since $\mathfrak{h}_\Lambda$ is the center of $\mathfrak{g}^s$, whence $\mathfrak{h}_\Lambda = g(V) \cap \mathfrak{h}$ and $g(V)$ is the direct sum of $\mathfrak{h}_\Lambda$ and $g(V) \cap \mathfrak{u}$. □

4.3. For x in $\mathfrak{g}$, denote by Z_x the subset of elements of $G.X$ containing x and by $(G^x)_0$ the identity component of G^x.

Lemma 4.6. *Let x be in $\mathfrak{N}_{\mathfrak{g}}$ and let Z be an irreducible component of Z_x. Suppose that some element of Z is not contained in $\mathfrak{N}_{\mathfrak{g}}$.*

(i) *For some torus $\mathfrak{s}$ of $\mathfrak{g}^x$, all element of a dense open subset of Z contains a conjugate of $\mathfrak{s}$ under $(G^x)_0$.*

(ii) *For some s in $\mathfrak{s}$ and for some irreducible component Z_1 of Z_{s+x}, Z is the closure in $\mathrm{Gr}_\ell(\mathfrak{g})$ of $(G^x)_0.Z_1$.*

(iii) *If Z_1 has dimension smaller than $\dim \mathfrak{g}^{s+x} - \ell$, then Z has dimension smaller than $\dim \mathfrak{g}^x - \ell$.*

Proof. (i) After some conjugation by an element of G, we can suppose that $\mathfrak{g}^x \cap \mathfrak{b}$ and $\mathfrak{g}^x \cap \mathfrak{h}$ are a Borel subalgebra and a maximal torus of $\mathfrak{g}^x$ respectively. Let Z_0 be the subset of elements of Z contained in $\mathfrak{b}$ and let $(B^x)_0$ be the identity component of B^x. Since Z is an irreducible component of Z_x, Z is invariant under $(G^x)_0$ and $Z = (G^x)_0.Z_0$. Since $(G^x)_0/(B^x)_0$ is a projective variety, according to the proof of Lemma 1.7, $(G^x)_0.Z_*$ is a closed subset of Z for all closed subset Z_* of Z. Hence for some irreducible component Z_* of Z_0, $Z = (G^x)_0.Z_*$.

For Λ subset of $\mathcal{R}$, denote by $Z_{*,\Lambda}$ the subset of elements V of Z_* such that

$$g(V) = \mathfrak{h}_\Lambda \oplus g(V) \cap \mathfrak{u}$$

for some g in $(B^x)_0$. According to Corollary 4.5, Z_* is the union of $Z_{*,\Lambda}$, $\Lambda \subset \mathcal{R}$. Since all element of $Z_{*,\Lambda}$ is contained in $\mathfrak{h}_\Lambda + \mathfrak{u}$,

$$\overline{Z_{*,\Lambda}} \subset \bigcup_{\mathcal{R}\supset\Lambda'\supset\Lambda} Z_{*,\Lambda'}.$$

So, by induction on $|\mathcal{R} \setminus \Lambda|$, $Z_{*,\Lambda}$ is a constructible subset of Z_*. Then, since $\mathcal{R}$ is finite, for some subset Λ of $\mathcal{R}$, $Z_{*,\Lambda}$ is dense in Z_*. As a result, $(G^x)_0.Z_{*,\Lambda}$ contains a dense open subset of Z and for all V in $(G^x)_0.Z_{*,\Lambda}$, the biggest torus contained in V is conjugate to $\mathfrak{h}_\Lambda$ under $(G^x)_0$.

(ii) For some s in $\mathfrak{s}$, $\mathfrak{g}^s$ is the centralizer of $\mathfrak{s}$ in $\mathfrak{g}$. Let Z^s be the subset of elements of Z containing s. Then Z^s is contained in Z_{s+x} and according to Corollary 4.3,(i), Z^s is the subset of elements of Z, containing $\mathfrak{s}$. By (i), for some irreducible component Z_1' of Z^s, $(G^x)_0.Z_1'$ is dense in Z. Let Z_1 be an irreducible component of Z_{s+x}, containing Z_1'. According to Corollary 4.3,(ii), Z_1 is contained in Z_x since x is the nilpotent component of $s + x$. So $Z_1 = Z_1'$ and $(G^x)_0.Z_1$ is dense in Z.

(iii) Since Z_1 is an irreducible component of Z_{s+x}, Z_1 is invariant under the identity component of G^{s+x}. Moreover, G^{s+x} is contained in G^x since x is the nilpotent component of $s + x$. As a result, by (ii),

$$\dim Z \leq \dim \mathfrak{g}^x - \dim \mathfrak{g}^{s+x} + \dim Z_1,$$

whence the assertion. □

Denote by C_h the G-invariant closed cone generated by h with h in $\mathfrak{h}$ such that $\beta(h) = 2$ for all β in Π.

Lemma 4.7. *Suppose $\mathfrak{g}$ semisimple. Let Γ be the closure in $\mathrm{Gr}_\ell(\mathfrak{g}) \times \mathfrak{g}$ of the image of the map*

$$\Bbbk^* \times G \longrightarrow \mathrm{Gr}_\ell(\mathfrak{g}) \times \mathfrak{g}\ , \qquad (t, g) \longmapsto (g(\mathfrak{h}), tg(h))$$

and Γ_0 the intersection of Γ and $\mathrm{Gr}_\ell(\mathfrak{g}) \times \mathfrak{N}_{\mathfrak{g}}$.

(i) *The subvariety Γ of $\mathrm{Gr}_\ell(\mathfrak{g}) \times \mathfrak{g}$ has dimension $2n + 1$. Moreover, Γ is contained in $\mathcal{E}$.*

(ii) *The varieties $G.X$ and C_h are the images of Γ by the first and second projections respectively.*

(iii) *The subvariety Γ_0 of Γ is equidimensional of codimension 1.*

(iv) *For x nilpotent in $\mathfrak{g}$, the subvariety of elements V of $G.X$, containing x and contained in $\overline{G(x)}$, has dimension at most $\dim \mathfrak{g}^x - \ell$.*

Proof. (i) Since the stabilizer of $(\mathfrak{h}, h)$ in $\Bbbk^* \times G$ equals $\{1\}\times H$, Γ has dimension $2n+1$. Since $tg(h)$ is in $g(\mathfrak{h})$ for all (t, g) in $\Bbbk^* \times G$ and $\mathcal{E}$ is a closed subset of $\mathrm{Gr}_\ell(\mathfrak{g}) \times \mathfrak{g}$, Γ is contained in $\mathcal{E}$.

(ii) Since $\mathrm{Gr}_\ell(\mathfrak{g})$ is a projective variety, the image of Γ by the second projection is closed in $\mathfrak{g}$. So, it equals C_h since it is contained in C_h and it contains the cone generated by $G.h$. Let Y be the image of Γ by the first projection. Since Γ is a closed subset of $\mathrm{Gr}_\ell(\mathfrak{g}) \times \mathfrak{g}$, invariant by the automorphisms $(V, x) \mapsto (V, tx)$ with t in $\Bbbk^*$, $Y \times \{0\}$ is the intersection of Γ and $\mathrm{Gr}_\ell(\mathfrak{g}) \times \{0\}$. Then Y is a closed subset of $\mathrm{Gr}_\ell(\mathfrak{g})$ containing $G.\mathfrak{h}$. Moreover Γ is contained in the closed subset $G.X\times\mathfrak{g}$ of $\mathrm{Gr}_\ell(\mathfrak{g}) \times \mathfrak{g}$. Hence $Y = G.X$.

(iii) The subvariety C_h of $\mathfrak{g}$ has dimension $2n + 1$ and the nullvariety of p_1 in C_h is contained in $\mathfrak{N}_{\mathfrak{g}}$ since it is the nullvariety in $\mathfrak{g}$ of the polynomials $p_1, \ldots, p_\ell$. Hence $\mathfrak{N}_{\mathfrak{g}}$ is the nullvariety of p_1 in C_h and Γ_0 is the nullvariety in Γ of the function $(V, x) \mapsto p_1(x)$. So Γ_0 is equidimensional of codimension 1 in Γ.

(iv) Let T be the subset of elements V of $G.X$, containing x and contained in $\overline{G(x)}$. Denote by Γ_T the inverse image of $\overline{G.T}$ by the projection $\Gamma \longrightarrow G.X$. Then Γ_T is contained in Γ_0. Since all element of T contains x and is contained in $\overline{G(x)}$ and since Γ_T is invariant under G, the image of Γ_T by the second projection is equal to $\overline{G(x)}$. Moreover, $T \times \{x\} \subset G.X \times \{x\} \cap \Gamma_T$. Hence

$$\dim \Gamma_T \geq \dim T + \dim \mathfrak{g} - \dim \mathfrak{g}^x.$$

By (i) and (iii),

$$\dim \Gamma_T \leq \dim \mathfrak{g} - \ell$$

since Γ_T is contained in Γ_0. Hence T has dimension at most $\dim \mathfrak{g}^x - \ell$. □

When $\mathfrak{g}$ is semisimple, denote by $(G.X)_{\mathrm{u}}$ the subset of elements of $G.X$ contained in $\mathfrak{N}_{\mathfrak{g}}$.

Corollary 4.8. *Suppose $\mathfrak{g}$ semisimple. Let x be in $\mathfrak{N}_{\mathfrak{g}}$.*
(i) *The variety $(G.X)_{\mathrm{u}}$ has dimension at most $2n - \ell$.*
(ii) *The variety $Z_x \cap (G.X)_{\mathrm{u}}$ has dimension at most $\dim \mathfrak{g}^x - \ell$.*

Proof. (i) Let T be an irreducible component of $(G.X)_{\mathrm{u}}$ and let $\mathcal{E}_T$ be the restriction to T of the vector bundle $\mathcal{E}$ over $G.X$. Then $\mathcal{E}_T$ is irreducible and has dimension $\dim T + \ell$. Denoting by Y the image of the projection $\mathcal{E}_T \longrightarrow \mathfrak{g}$, Y is an irreducible closed subvariety of $\mathfrak{g}$ contained in $\mathfrak{N}_{\mathfrak{g}}$. The subvariety $(G.X)_{\mathrm{u}}$ of $G.X$ is invariant under G since so is $\mathfrak{N}_{\mathfrak{g}}$. Hence $\mathcal{E}_T$ and Y are G-invariant and for some y in $\mathfrak{N}_{\mathfrak{g}}$, $Y = \overline{G(y)}$ since $\mathfrak{N}_{\mathfrak{g}}$ is a finite union of orbits. Denoting by F_y the fiber at y of the projection $\mathcal{E}_T \longrightarrow Y$, V is contained in $\overline{G(y)}$ and contains y for all V in F_y. So, by Lemma 4.7,(iv),

$$\dim F_y \leq \dim \mathfrak{g}^y - \ell.$$

Since the projection is G-equivariant, this inequality holds for the fibers at the elements of $G(y)$. Hence,

$$\dim \mathcal{E}_T \leq \dim \mathfrak{g} - \ell \text{ and } \dim T \leq 2n - \ell.$$

(ii) Let Z be an irreducible component of $Z_x \cap (G.X)_{\mathrm{u}}$ and let T be an irreducible component of $(G.X)_{\mathrm{u}}$, containing Z. Let $\mathcal{E}_T$ and Y be as in (i). Then $G(x)$ is contained in Y and the inverse image of $\overline{G(x)}$ in $\mathcal{E}_T$ has dimension at least $\dim G(x) + \dim Z$. So, by (i),

$$\dim G(x) + \dim Z \leq \dim \mathfrak{g} - \ell,$$

whence the assertion. □

Theorem 4.9. *For x in $\mathfrak{g}$, the variety of elements of $G.X$, containing x, has dimension at most $\dim \mathfrak{g}^x - \ell$.*

Proof. Prove the theorem by induction on $\dim \mathfrak{g}$. If $\mathfrak{g}$ is commutative, $G.X = \{\mathfrak{g}\}$. If the derived Lie algebra of $\mathfrak{g}$ is simple of dimension 3, $G.X$ has dimension 2 and for x not in the center of $\mathfrak{g}$, $Z_x = \{\mathfrak{g}^x\}$. Suppose the theorem true for all reductive Lie algebra of dimension strictly smaller than $\dim \mathfrak{g}$. Let x be in $\mathfrak{g}$. Since $G.X$ has dimension $\dim \mathfrak{g} - \ell$, we can suppose that x is not in the center of $\mathfrak{g}$. Suppose that x is not nilpotent. Then $\mathfrak{g}^{x_{\mathrm{s}}}$ has dimension strictly smaller than $\dim \mathfrak{g}$ and all element of $G.X$ containing x is contained in $\mathfrak{g}^{x_{\mathrm{s}}}$ and contains the center of $\mathfrak{g}^{x_{\mathrm{s}}}$ by Corollary 4.3,(i). So, by Lemma 4.4,(ii), Z_x is contained in $\overline{G^{x_{\mathrm{s}}}.\mathfrak{h}}$, whence the theorem in this case by induction hypothesis. As a result, by Lemma 4.6, for all x in $\mathfrak{g}$, all irreducible component of Z_x, containing an element not contained in $\mathfrak{N}_{\mathfrak{g}}$, has dimension at most $\dim \mathfrak{g}^x - \ell$.

Let x be a nilpotent element of $\mathfrak{g}$. Denoting by Z'_x the subset of elements of $\overline{G.(\mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}])}$ containing x, Z_x is the image of Z'_x by the map $V \mapsto V + \mathfrak{z}_{\mathfrak{g}}$, whence the theorem by Corollary 4.8. □

4.4. Let s be in $\mathfrak{h}\setminus\{0\}$. Set $\mathfrak{p} := \mathfrak{g}^s + \mathfrak{b}$ and denote by $\mathfrak{p}_{\mathrm{u}}$ the nilpotent radical of $\mathfrak{p}$. Let P be the normalizer of $\mathfrak{p}$ in G and let P_{u} be its unipotent radical. For a nilpotent orbit Ω of G^s in $\mathfrak{g}^s$, denote by $\Omega^{\#}$ the induced orbit by Ω from $\mathfrak{g}^s$ to $\mathfrak{g}$.

Lemma 4.10. *Let Y be a G-invariant irreducible closed subset of $\mathfrak{g}$ and let Y' be the union of G-orbits of maximal dimension in Y. Suppose that s is the semisimple component of an element x of Y'. Denote by Ω the orbit of x_{n} under G^s and set $Y_1 := \mathfrak{z}_s + \overline{\Omega} + \mathfrak{p}_{\mathrm{u}}$.*

(i) *The subset Y_1 of $\mathfrak{p}$ is closed and invariant under P.*

(ii) *The subset $G(Y_1)$ of $\mathfrak{g}$ is a closed subset of dimension $\dim \mathfrak{z}_s + \dim G(x)$.*

(iii) *For some nonempty open subset Y'' of Y', the conjugacy class of $\mathfrak{g}^{y_{\mathrm{s}}}$ under G does not depend on the element y of Y''.*

(iv) *For a good choice of x in Y'', Y is contained in $G(Y_1)$.*

Proof. (i) By [Ko63, §3.2, Lemma 5], G^s is connected and $P = P_{\mathrm{u}}G^s$. For all y in $\mathfrak{p}$ and for all g in P_{u}, $g(y)$ is in $y + \mathfrak{p}_{\mathrm{u}}$. Hence Y_1 is invariant under P since it is invariant under G^s. Moreover, it is a closed subset of $\mathfrak{p}$ since $\mathfrak{z}_s + \overline{\Omega}$ is a closed subset of $\mathfrak{g}^s$.

(ii) According to (i) and Lemma 1.7, $G(Y_1)$ is a closed subset of $\mathfrak{g}$. According to [CMa93, Theorem 7.1.1], $\Omega^{\#}\cap(\Omega+\mathfrak{p}_{\mathrm{u}})$ is a P-orbit and the centralizers in $\mathfrak{g}$ of its elements are contained in $\mathfrak{p}$. For y in $\Omega^{\#}\cap(\Omega+\mathfrak{p}_{\mathrm{u}})$ and for g in G, if $g(y)$ is in Y_1 then it is in $\Omega+\mathfrak{p}_{\mathrm{u}}$ since it is nilpotent. So, for y in $\Omega^{\#}\cap(\Omega+\mathfrak{p}_{\mathrm{u}})$, the subset of elements g of G such that $g(y)$ is in Y_1 has dimension $\dim\mathfrak{p}$. As a result,

$$\dim G(Y_1) = \dim G\times_P Y_1 = \dim \mathfrak{p}_{\mathrm{u}} + \dim Y_1 .$$

Since $\dim\mathfrak{g}^x = \dim\mathfrak{g}^s - \dim\Omega$,

$$\begin{aligned}\dim Y_1 &= \dim\mathfrak{z}_s + \dim\mathfrak{p}_{\mathrm{u}} + \dim\mathfrak{g}^s - \dim\mathfrak{g}^x\\ \dim G(Y_1) &= \dim\mathfrak{z}_s + 2\dim\mathfrak{p}_{\mathrm{u}} + \dim\mathfrak{g}^s - \dim\mathfrak{g}^x\\ &= \dim\mathfrak{z}_s + \dim G(x).\end{aligned}$$

(iii) Let τ be the canonical morphism from $\mathfrak{g}$ to its categorical quotient $\mathfrak{g}//G$ under G and let Z be the closure in $\mathfrak{g}//G$ of $\tau(Y)$. Since Y is irreducible, Z is irreducible and there exists an irreducible component $\widetilde{Z}$ of the preimage of Z in $\mathfrak{h}$ whose image in $\mathfrak{g}//G$ equals Z. Since the set of conjugacy classes under G of the centralizers of the elements of $\mathfrak{h}$ in $\mathfrak{g}$ is finite, for some nonempty open subset $Z^{\#}$ of $\widetilde{Z}$, the centralizers of its elements are conjugate under G. The image of $Z^{\#}$ in $\mathfrak{g}//G$ contains a dense open subset Z' of Z. Let Y'' be the inverse image of Z' by the restriction of τ to Y'. Then Y'' is a dense open subset of Y and the centralizers in $\mathfrak{g}$ of the semisimple components of its elements are conjugate under G.

(iv) Suppose that x is in Y''. Let Z_Y be the set of elements y of Y'' such that $\mathfrak{g}^{y_{\mathrm{s}}} = \mathfrak{g}^s$. Then $G.Z_Y = Y''$. For all nilpotent orbit Ω of G^s in $\mathfrak{g}^s$, set:

$$Y_\Omega = \mathfrak{z}_s + \overline{\Omega} + \mathfrak{p}_{\mathrm{u}}$$

Then Z_Y is contained in the union of the Y_Ω's. Hence Y'' is contained in the union of the $G(Y_\Omega)$'s. According to (ii), $G(Y_\Omega)$ is a closed subset of $\mathfrak{g}$. Hence Y is contained in the union of the $G(Y_\Omega)$'s since Y'' is dense in Y. Then Y is contained in $G(Y_\Omega)$ for some Ω since Y is irreducible and there are finitely many nilpotent orbits in $\mathfrak{g}^s$, whence the assertion. □

Theorem 4.11. *(i) The variety $G.X$ is the union of $G.\mathfrak{h}$ and the $G.X_\beta$'s, $\beta\in\Pi$.*

(ii) *The variety X is the union of $U.\mathfrak{h}$ and the X_α's, $\alpha\in\mathcal{R}_+$.*

Proof. Let μ be the map

$$\mathrm{Gr}_{\ell'}([\mathfrak{g},\mathfrak{g}]) \longrightarrow \mathrm{Gr}_{\ell}(\mathfrak{g}) , \qquad V \longmapsto \mathfrak{z}_{\mathfrak{g}} + V$$

with ℓ' the rank of $[\mathfrak{g},\mathfrak{g}]$ and set:

$$X_d := \overline{B.(\mathfrak{h}\cap[\mathfrak{g},\mathfrak{g}])}, \qquad X_{\alpha,d} := \overline{B.(V_\alpha\cap[\mathfrak{g},\mathfrak{g}])}$$

for α in $\mathcal{R}_+$. Then X, $G.X$, X_α, $G.X_\alpha$ are the images of X_d, $G.X_d$, $X_{\alpha,d}$, $G.X_{\alpha,d}$ by μ respectively. So we can suppose $\mathfrak{g}$ semisimple.

(i) For $\ell = 1$, $\mathfrak{g}$ is simple of dimension 3. In this case, $G.X$ is the union of $G.\mathfrak{h}$ and $G.\mathfrak{g}^e$. So, we can suppose $\ell \geq 2$. According to Lemma 4.1,(iii), for α in $\mathcal{R}_+$, $G.X_\alpha$ is an irreducible component of $G.X \setminus G.\mathfrak{h}$. Moreover, for all β in $\Pi \cap W(\mathcal{R})(\alpha)$, $G.X_\alpha = G.X_\beta$ since V_α and V_β are conjugate under $N_G(\mathfrak{h})$.

Let T be an irreducible component of $G.X \setminus G.\mathfrak{h}$. Set:

$$\mathcal{E}_T := \mathcal{E} \cap T \times \mathfrak{g}$$

and denote by Y the image of $\mathcal{E}_T$ by the second projection. Then Y is closed in $\mathfrak{g}$ since $\mathrm{Gr}_\ell(\mathfrak{g})$ is a projective variety. Since $\mathcal{E}_T$ is a vector bundle over T and since T is irreducible, $\mathcal{E}_T$ is irreducible and so is Y. Since T is an irreducible component of $G.X \setminus G.\mathfrak{h}$, T, $\mathcal{E}_T$ and Y are G-invariant. According to Lemma 4.1,(iii), T has codimension 1 in $G.X$. Hence, by Corollary 4.8,(i) Y is not contained in the nilpotent cone since $\ell \geq 2$. Let Y' be the set of elements x of Y such that $\mathfrak{g}^x$ has minimal dimension. According to Lemma 4.10,(ii) and (iv), for some x in Y',

$$\dim Y \leq \dim G(x) + \dim \mathfrak{z}_{x_{\mathrm{s}}}$$

and according to Theorem 4.9,

$$\dim \mathcal{E}_T \leq \dim G(x) + \dim \mathfrak{z}_{x_{\mathrm{s}}} + \dim \mathfrak{g}^x - \ell = \dim \mathfrak{g} + \dim \mathfrak{z}_{x_{\mathrm{s}}} - \ell$$

Hence $\mathcal{E}_T$ has dimension at most $2n + \dim \mathfrak{z}_{x_{\mathrm{s}}}$ and $\dim \mathfrak{z}_{x_{\mathrm{s}}} = \ell - 1$ since T has codimension 1 in $G.X$. As a result, x_{s} is subregular and for some g in G, $g(\mathfrak{z}_{x_{\mathrm{s}}})$ is the kernel of a positive root α. Denoting by $\mathfrak{s}_\alpha$ the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{-\alpha}$, $\mathfrak{g}^{g(x_{\mathrm{s}})}$ is the direct sum of $\mathfrak{h}_\alpha$ and $\mathfrak{s}_\alpha$. Since the maximal commutative subalgebras of $\mathfrak{s}_\alpha$ have dimension 1, a commutative subalgebra of dimension ℓ of $\mathfrak{g}^{g(x_{\mathrm{s}})}$ is either a Cartan subalgebra of $\mathfrak{g}$ or conjugate to V_α under the adjoint group of $\mathfrak{g}^{g(x_{\mathrm{s}})}$. As a result, V_α is in T and $T = \overline{G.V_\alpha} = G.X_\alpha$ since T is G-invariant, whence the assertion.

(ii) According to Lemma 4.1,(ii), for α in $\mathcal{R}_+$, X_α is an irreducible component of $X \setminus B.\mathfrak{h}$. Let $\mathfrak{g}_1,\ldots,\mathfrak{g}_m$ be the simple factors of $\mathfrak{g}$. For $j = 1,\ldots,m$, denote by X_j the closure in $\mathrm{Gr}_{\mathrm{rk}\,\mathfrak{g}_j}(\mathfrak{g}_j)$ of the orbit of $\mathfrak{h}\cap\mathfrak{g}_j$. Then $X = X_1\times\cdots\times X_m$ and the complement to $B.\mathfrak{h}$ in X is the union of the

$$X_1\times\cdots\times X_{j-1}\times(X_j\setminus B.(\mathfrak{h}\cap\mathfrak{g}_j))\times X_{j+1}\times\cdots\times X_m$$

So, we can suppose $\mathfrak{g}$ simple. Consider

$$\mathfrak{b} = \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_\ell = \mathfrak{g}$$

an increasing sequence of parabolic subalgebras verifying the following condition: for $i = 0,\ldots,\ell-1$, there is no parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ such that

$$\mathfrak{p}_i \subsetneq \mathfrak{q} \subsetneq \mathfrak{p}_{i+1}.$$

For $i = 0,\ldots,\ell$, let P_i be the normalizer of $\mathfrak{p}_i$ in G, let $\mathfrak{p}_{i,\mathrm{u}}$ be the nilpotent radical of $\mathfrak{p}_i$ and let $P_{i,\mathrm{u}}$ be the unipotent radical of P_i. For $i = 0,\ldots,\ell$ and for α in $\mathcal{R}_+$, set $X_i := P_i.X$ and $X_{i,\alpha} := P_i.X_\alpha$.

Prove by induction on $\ell - i$ that for all sequence of parabolic subalgebras verifying the above condition, the $X_{i,\alpha}$'s, $\alpha \in \mathcal{R}_+$, are the irreducible components of $X_i \setminus P_i.\mathfrak{h}$.

For $i = \ell$, it results from (i). Suppose that it is true for $i + 1$. According to Lemma 4.1,(iii), the $X_{i,\alpha}$'s are irreducible components of $X_i \setminus P_i.\mathfrak{h}$.

Claim 4.12. Let T be an irreducible component of $X_i \setminus P_i.\mathfrak{h}$ such that P_i is its stabilizer in P_{i+1}. Then $T = X_{i,\alpha}$ for some α in $\mathcal{R}_+$.

Proof. According to the induction hypothesis, T is contained in $X_{i+1,\alpha}$ for some α in $\mathcal{R}_+$. According to Lemma 4.1,(iv), T has codimension 1 in X_i so that $P_{i+1}.T$ and $X_{i+1,\alpha}$ have the same dimension. Then they are equal and T contains $\mathfrak{g}^x$ for some x in $\mathfrak{b}_{\rm reg}$ such that $x_{\rm s}$ is a subregular element belonging to $\mathfrak{h}$. Denoting by α' the positive root such that $\alpha'(x_{\rm s}) = 0$, $\mathfrak{g}^x = V_{\alpha'}$ since $V_{\alpha'}$ is the commutative subalgebra contained in $\mathfrak{b}$ and containing $\mathfrak{h}_{\alpha'}$, which is not Cartan, so that $T = X_{i,\alpha'}$. □

Suppose that $X_i \setminus P_i.\mathfrak{h}$ is not the union of the $X_{i,\alpha}$'s, $\alpha \in \mathcal{R}_+$. We expect a contradiction. Let T be an irreducible component of $X_i \setminus P_i.\mathfrak{h}$, different from $X_{i,\alpha}$ for all α. According to Claim 4.12 and according to the condition verified by the sequence, T is invariant under P_{i+1}. Moreover, according to Claim 4.12, it is so for all sequence $\mathfrak{p}'_0, \ldots, \mathfrak{p}'_\ell$ of parabolic subalgebras verifying the above condition and such that $\mathfrak{p}'_j = \mathfrak{p}_j$ for $j = 0, \ldots, i$. As a result, for all simple root β such that $\mathfrak{g}^{-\beta}$ is not in $\mathfrak{p}_i$, T is invariant under the one parameter subgroup of G generated by ${\rm ad}\,\mathfrak{g}^{-\beta}$. Hence T is invariant under G. It is impossible since for x in $\mathfrak{g} \setminus \{0\}$, the orbit $G(x)$ is not contained in $\mathfrak{p}_i$ since $\mathfrak{g}$ is simple, whence the assertion. □

4.5. Let X' be the subset of $\mathfrak{g}^x$ with x in $\mathfrak{b}_{\rm reg}$ such that $x_{\rm s}$ is regular or subregular. For α in $\mathcal{R}_+$, denote by θ_α the map

$$\Bbbk \longrightarrow X, \qquad t \longmapsto \exp(t\,{\rm ad}\,x_\alpha).\mathfrak{h}.$$

According to [Sh94, Ch. VI, Theorem 1], θ_α has a regular extension to $\mathbb{P}^1(\Bbbk)$, also denoted by θ_α. Set $Z_\alpha := \theta_\alpha(\mathbb{P}^1(\Bbbk))$ and $X'_\alpha := B.Z_\alpha$ so that $X'_\alpha = U.\mathfrak{h} \cup B.V_\alpha$.

Lemma 4.13. *Let α be in $\mathcal{R}_+$ and let V be in X. Then V is in $B.Z_\alpha$ if and only if $g(V)$ contains $\mathfrak{h}_\alpha$ for some g in B.*

Proof. The condition is necessary by definition. Suppose that V contains $\mathfrak{h}_\alpha$. Since V is commutative by Corollary 4.3,(ii), V is contained in $\mathfrak{h} + \mathfrak{g}^\alpha$. If V is a Cartan subalgebra, then $V = \theta_\alpha(t)$ for some t in $\Bbbk$. Otherwise, $V = \theta_\alpha(\infty)$, whence the lemma. □

Corollary 4.14. *Let α be a positive root.*

(i) *The sets X'_α and $G.X'_\alpha$ are open subsets of X and $G.X$ respectively.*

(ii) *The sets X' and $G.X'$ are big open subsets of X and $G.X$ respectively.*

Proof. (i) Since X'_α is a B-invariant subset containing the open subset $U.\mathfrak{h}$, it suffices to prove that X'_α is a neighborhood of V_α in X. Denote by H_α the coroot of α and set:

$$E' := \bigoplus_{\gamma \in \mathcal{R}_+ \setminus \{\alpha\}} \mathfrak{g}^\gamma, \qquad E := \Bbbk H_\alpha \oplus E'.$$

Let Ω_E be the set of subspaces V of $\mathfrak{b}$ such that E is a complement to V in $\mathfrak{b}$ and let Ω'_E be the complement in $X \cap \Omega_E$ to the union of $X_\gamma, \gamma \in \mathcal{R}_+ \setminus \{\alpha\}$. Then Ω'_E is an open neighborhood of V_α in X. Since X'_α contains $U.\mathfrak{h}$, X'_α contains all the Cartan subalgebras contained in Ω'_E. Let V be in Ω'_E such that V is not a Cartan subalgebra. According to Corollary 4.5, for some nonempty subset

Λ of $\mathcal{R}$, V is contained $\mathfrak{h}_\Lambda + \mathfrak{u}$ and contains a conjugate of $\mathfrak{h}_\Lambda$ under B. Then $\mathfrak{h} = \Bbbk H_\alpha + \mathfrak{h}_\Lambda$ since V is in Ω_E. As a result, $\mathfrak{h}_\Lambda = \mathfrak{h}_\gamma$ for some positive root γ and V is conjugate to V_γ under B by Lemma 4.13. Since V is not in X_δ for all δ in $\mathcal{R}_+ \setminus \{\alpha\}$, $\gamma = \alpha$ and V is in X'_α. Then X'_α contains Ω'_E. As a result, X'_α is an open subset of X and $G.(X \setminus X'_\alpha)$ is a closed subset of $G.X$ by Lemma 1.7, whence the assertion.

(ii) By definition, X' is the union of $X'_\alpha, \alpha \in \mathcal{R}_+$. Hence X' is an open subset of X by (i). Moreover, by Theorem 4.11,(ii), $X \setminus X'$ is the union of $X_\alpha \setminus X', \alpha \in \mathcal{R}_+$. Then X' is a big open subset of X since, for all α, $X_\alpha \setminus X'$ is strictly contained in the irreducible subvariety X_α of X.

Since $G.X'$ is the union of $G.X'_\alpha, \alpha \in \mathcal{R}_+$, $G.X'$ is an open subset of $G.X$ by (i). Moreover, by Theorem 4.11,(i), $G.X \setminus G.X'$ is the union of $G.X_\beta \setminus G.X', \beta \in \Pi$. Hence $G.X'$ is a big open subset of $G.X$ since, for all β, $G.X_\beta \setminus G.X'$ is strictly contained in the irreducible subvariety $G.X_\beta$ of $G.X$. □

Proposition 4.15. *The sets X' and $G.X'$ are smooth big open subsets of X and $G.X$ respectively.*

Proof. According to Corollary 4.14,(ii), it remains to prove that X' and $G.X'$ are smooth open subsets of X and $G.X$ respectively. Denote by π the bundle projection of the vector bundle $\mathcal{E}$ over $G.X$. Recall $\mathcal{E}_0 := \pi^{-1}(X)$. Let μ be the map

$$\mathfrak{g}_{\rm reg} \longrightarrow \mathrm{Gr}_\ell(\mathfrak{g}) , \qquad x \longmapsto \mathfrak{g}^x$$

and let μ_0 be its restriction to $\mathfrak{b}_{\rm reg}$. Then μ is a regular map. Let Γ_μ and Γ_{μ_0} be the images of the graphs of μ and μ_0 respectively by the isomorphism

$$\mathfrak{g} \times \mathrm{Gr}_\ell(\mathfrak{g}) \longrightarrow \mathrm{Gr}_\ell(\mathfrak{g}) \times \mathfrak{g} , \qquad (x, V) \longmapsto (V, x).$$

Then Γ_μ and Γ_{μ_0} are smooth varieties contained in $\mathcal{E}$ and $\mathcal{E}_0$ respectively since for x in $\mathfrak{g}_{\rm reg,ss}$, $\mathfrak{g}^x$ is a Cartan subalgebra, contained in $\mathfrak{b}$ when x is in $\mathfrak{b}$. Set:

$$\Gamma'_\mu := \Gamma_\mu \cap \pi^{-1}(G.X') = \mathcal{E} \cap G.X' \times \mathfrak{g}_{\rm reg} \quad \text{and} \quad \Gamma'_{\mu_0} := \Gamma_{\mu_0} \cap \pi^{-1}(X') = \mathcal{E} \cap X' \times \mathfrak{b}_{\rm reg}.$$

Then Γ'_μ is a smooth variety as an open susbet of Γ_μ and Γ'_μ is an open subset of $\pi^{-1}(G.X')$ such that $\pi(\Gamma'_\mu) = G.X'$ since all element of $G.X'$ contains regular elements. In the same way, Γ'_{μ_0} is a smooth open subset of $\pi^{-1}(X')$ such that $\pi(\Gamma'_{\mu_0}) = X'$. As a result, Γ'_μ and Γ'_{μ_0} are smooth open subsets of vector bundles over $G.X'$ and X' respectively since $\mathcal{E}$ and $\mathcal{E}_0$ are vector bundles over $G.X$ and X respectively. Hence $G.X'$ and X' are smooth varieties by [MA86, Ch. 8, Theorem 23.7]. □

Summarizing the results of the section, Theorem 1.2,(i) is given by Corollay 4.3,(ii), Theorem 1.2,(ii) is given by Theorem 4.9, Theorem 1.2,(iii) is given by Lemma 4.1,(iv) since X and $G.X$ have dimension n and $2n$ respectively and Theorem 1.2,(iv) is given by Proposition 4.15.

5. On the generalized isospectral commuting variety

Let $k \geq 2$ be an integer. According to Section 2, we have the commutative diagram

$$\begin{tikzcd} G \times_B \mathfrak{b}^k \arrow[rr, "\gamma_{\rm x}"] \arrow[dr, "\gamma"'] & & \mathcal{B}_{\rm x}^{(k)} \arrow[dl, "\eta"] \\ & \mathcal{B}^{(k)} & \end{tikzcd} .$$

By Lemma 2.7,(i), ι_k is a closed embedding of $\mathfrak{b}^k$ into $\mathcal{B}_{\rm x}^{(k)}$, by Corollary 2.8,(i) $\mathcal{B}_{\rm x}^{(k)} = G.\iota_k(\mathfrak{b}^k)$ is closed in $\mathcal{X}^k$ and η is the restriction to $\mathcal{B}_{\rm x}^{(k)}$ of the canonical projection from $\mathcal{X}^k$ to $\mathfrak{g}^k$. Denote by $\mathcal{C}^{(k)}$ the closure of $G.\mathfrak{h}^k$ in $\mathfrak{g}^k$ with respect to the diagonal action of G in $\mathfrak{g}^k$ and set $\mathcal{C}_{\rm x}^{(k)} := \eta^{-1}(\mathcal{C}^{(k)})$.

The varieties $\mathcal{C}^{(k)}$ and $\mathcal{C}_{\mathrm{x}}^{(k)}$ are called *generalized commuting variety* and *generalized isospectral commuting variety* respectively. For $k=2$, $\mathcal{C}_{\mathrm{x}}^{(k)}$ is the isospectral commuting variety considered by M. Haiman in [Ha99, §8] and [Ha02, §7.2].

5.1. Set:

$$\mathcal{E}_0^{(k)} := \{(u, x_1, \ldots, x_k) \in X \times \mathfrak{b}^k \mid u \ni x_1, \ldots, u \ni x_k\}.$$

Lemma 5.1. *Denote by $\mathcal{E}_0^{(k,*)}$ the intersection of $\mathcal{E}_0^{(k)}$ and $U.\mathfrak{h} \times (\mathfrak{g}_{\mathrm{reg,ss}} \cap \mathfrak{b})^k$ and for w in $W(\mathcal{R})$, denote by θ_w the map*

$$\mathcal{E}_0^{(k)} \longrightarrow \mathfrak{b}^k \times \mathfrak{h}^k, \qquad (u, x_1, \ldots, x_k) \longmapsto (x_1, \ldots, x_k, w(\overline{x_1}), \ldots, w(\overline{x_k})).$$

(i) *Denoting by $\mathfrak{X}_{0,k}$ the image of $\mathcal{E}_0^{(k)}$ by the projection $(u, x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k)$, $\mathfrak{X}_{0,k}$ is the closure of $B.\mathfrak{h}^k$ in $\mathfrak{b}^k$ and $\mathcal{C}^{(k)}$ is the image of $G \times \mathfrak{X}_{0,k}$ by the map $(g, x_1, \ldots, x_k) \mapsto (g(x_1), \ldots, g(x_k))$.*

(ii) *For all w in $W(\mathcal{R})$, $\theta_w(\mathcal{E}_0^{(k,*)})$ is dense in $\theta_w(\mathcal{E}_0^{(k)})$.*

Proof. (i) Since X is a projective variety, $\mathfrak{X}_{0,k}$ is a closed subset of $\mathfrak{b}^k$. The variety $\mathcal{E}_0^{(k)}$ is irreducible of dimension $n + k\ell$ as a vector bundle of rank $k\ell$ over the irreducible variety X. So, $B.(\{\mathfrak{h}\} \times \mathfrak{h}^k)$ is dense in $\mathcal{E}_0^{(k)}$ and $\mathfrak{X}_{0,k}$ is the closure of $B.\mathfrak{h}^k$ in $\mathfrak{b}^k$, whence the assertion by Lemma 1.7.

(ii) Since $U.\mathfrak{h} \times (\mathfrak{g}_{\mathrm{reg,ss}} \cap \mathfrak{b})^k$ is an open susbet of $X \times \mathfrak{b}^k$, $\mathcal{E}_0^{(k,*)}$ is an open subset of $\mathcal{E}_0^{(k)}$. Moreover, it is a dense open subset since $\mathcal{E}_0^{(k)}$ is irreducible, whence the assertion since θ_w is a morphism of algebraic varieties. □

5.2. Let s be in $\mathfrak{h}$. According to [Ko63, §3.2, Lemma 5], G^s is connected. Denote by $\mathcal{R}_s$ the set of roots whose kernel contains s and denote by $W(\mathcal{R}_s)$ the Weyl group of $\mathcal{R}_s$.

Lemma 5.2. *Let $x = (x_1, \ldots, x_k)$ be in $\mathcal{C}^{(k)}$ verifying the following conditions:*

(1) *s is the semisimple component of x_1,*
(2) *for z in E_x, the centralizer in $\mathfrak{g}$ of the semisimple component of z has dimension at least* $\dim \mathfrak{g}^s$.

Then for $i = 1, \ldots, k$, the semisimple component of x_i is in $\mathfrak{z}_s$.

Proof. Since x is in $\mathcal{C}^{(k)}$, $[x_i, x_j] = 0$ for all (i, j). Suppose that for some i, the semisimple component $x_{i,\mathrm{s}}$ of x_i is not in $\mathfrak{z}_s$. A contradiction is expected. Since $[x_1, x_i] = 0$, for all t in $\Bbbk$, $s + tx_{i,\mathrm{s}}$ is the semisimple component of $x_1 + tx_i$. Moreover, after conjugation by an element of G^s, we can suppose that $x_{i,\mathrm{s}}$ is in $\mathfrak{h}$. Since $\mathcal{R}$ is finite, there exists t in $\Bbbk^*$ such that the subset of roots whose kernel contains $s + tx_{i,\mathrm{s}}$ is contained in $\mathcal{R}_s$. Since $x_{i,\mathrm{s}}$ is not in $\mathfrak{z}_s$, for some α in $\mathcal{R}_s$, $\alpha(s + tx_{i,\mathrm{s}}) \neq 0$ that is $\mathfrak{g}^{s+tx_{i,\mathrm{s}}}$ is strictly contained in $\mathfrak{g}^s$, whence the contradiction. □

For w in $W(\mathcal{R})$, set:

$$C_w := G^s wB/B, \qquad B^w := wBw^{-1}.$$

Lemma 5.3. [Hu95, §6.17, Lemma] *Let $\mathfrak{B}$ be the set of Borel subalgebras of $\mathfrak{g}$ and let $\mathfrak{B}_s$ be the set of Borel subalgebras of $\mathfrak{g}$ containing s.*

(i) *For all w in $W(\mathcal{R})$, C_w is a connected component of $\mathfrak{B}_s$.*

(ii) *For (w, w') in $W(\mathcal{R}) \times W(\mathcal{R})$, $C_w = C_{w'}$ if and only if $w'w^{-1}$ is in $W(\mathcal{R}_s)$.*

(iii) *The variety C_w is isomorphic to $G^s/(G^s \cap B^w)$.*

For x in $\mathcal{B}^{(k)}$, denote by $\mathfrak{B}_x$ the subset of Borel subalgebras containing E_x.

Corollary 5.4. *Let $x = (x_1, \dots, x_k)$ be in $\mathcal{C}^{(k)}$. Suppose that x verifies Conditions* (1) *and* (2) *of Lemma 5.2. Then $\{C_w \cap \mathfrak{B}_x \mid w \in W(\mathcal{R})\}$ is the set of connected components of $\mathfrak{B}_x$.*

Proof. Since a Borel subalgebra contains the semisimple component of its elements and since s is the semisimple component of x_1, $\mathfrak{B}_x$ is contained in $\mathfrak{B}_s$. As a result, according to Lemma 5.3,(i), every connected component of $\mathfrak{B}_x$ is contained in C_w for some w in $W(\mathcal{R})$. Set $x_{\rm n} := (x_{1,\rm n}, \dots, x_{k,\rm n})$. Since $[x_i, x_j] = 0$ for all (i, j), E_x is contained in $\mathfrak{g}^s$. Let $\mathfrak{B}^s$ be the set of Borel subalgebras of $\mathfrak{g}^s$ and for y in $(\mathfrak{g}^s)^k$, let $\mathfrak{B}^s_y$ be the set of Borel subalgebras of $\mathfrak{g}^s$ containing E_y. According to [Hu95, Theorem 6.5], $\mathfrak{B}^s_{x_{\rm n}}$ is connected. Moreover, according to Lemma 5.2, the semisimple components of $x_1, \dots, x_k$ are in $\mathfrak{z}_s$ so that $\mathfrak{B}^s_{x_{\rm n}} = \mathfrak{B}^s_x$. Let w be in $W(\mathcal{R})$. According to Lemma 5.3,(iii), there is an isomorphism from $\mathfrak{B}^s$ to C_w. Moreover, the image of $\mathfrak{B}^s_x$ by this isomorphism is equal to $C_w \cap \mathfrak{B}_x$, whence the corollary. □

Corollary 5.5. *Let $x = (x_1, \dots, x_k)$ be in $\mathcal{C}^{(k)}$ verifying Conditions* (1) *and* (2) *of Lemma 5.2. Then $\eta^{-1}(x)$ is contained in $\{(x_1, \dots, x_k, w(x_{1,\rm s}), \dots, w(x_{k,\rm s})) \mid w \in W(\mathcal{R})\}$.*

Proof. Since $\gamma = \eta \circ \gamma_{\rm x}$, $\eta^{-1}(x)$ is the image of $\gamma^{-1}(x)$ by $\gamma_{\rm x}$. Furthermore, $\gamma_{\rm x}$ is constant on the connected components of $\gamma^{-1}(x)$ since $\eta^{-1}(x)$ is finite. Let C be a connected component of $\gamma^{-1}(x)$. Identifying $G \times_B \mathfrak{b}^k$ with the subvariety of elements (u, x) of $\mathfrak{B} \times \mathfrak{g}^k$ such that E_x is contained in u, C identifies with $(C_w \cap \mathfrak{B}_x) \times \{x\}$ for some w in $W(\mathcal{R})$ by Corollary 5.4. Then for some g in G^s and for some representative g_w of w in $N_G(\mathfrak{h})$, $gg_w(\mathfrak{b})$ contains E_x so that

$$\gamma_{\rm x}(C) = \{(x_1, \dots, x_k, \overline{(gg_w)^{-1}(x_1)}, \dots, \overline{(gg_w)^{-1}(x_k)})\}.$$

By Lemma 5.2, $x_{1,\rm s}, \dots, x_{k,\rm s}$ are in $\mathfrak{z}_s$ so that $w^{-1}(x_{i,\rm s})$ is the semisimple component of $(gg_w)^{-1}(x_i)$ for $i = 1, \dots, k$. Hence

$$\gamma_{\rm x}(C) = \{(x_1, \dots, x_k, w^{-1}(x_{1,\rm s}), \dots, w^{-1}(x_{k,\rm s}))\},$$

whence the corollary. □

Proposition 5.6. *The variety $\mathcal{C}^{(k)}_{\rm x}$ is irreducible and equal to the closure of $G.\iota_k(\mathfrak{h}^k)$ in $\mathcal{B}^{(k)}_{\rm x}$.*

Proof. Denote by $\overline{G.\iota_k(\mathfrak{h}^k)}$ the closure of $G.\iota_k(\mathfrak{h}^k)$ in $\mathcal{B}^{(k)}_{\rm x}$. Then $\overline{G.\iota_k(\mathfrak{h}^k)}$ is irreducible as the closure of an irreducible set. Since η is G-equivariant, $\eta(G.\iota_k(\mathfrak{h}^k)) = G.\mathfrak{h}^k$. Hence $\eta(\overline{G.\iota_k(\mathfrak{h}^k)}) = \mathcal{C}^{(k)}$ since η is a finite morphism and $\mathcal{C}^{(k)}$ is the closure of $G.\mathfrak{h}^k$ in $\mathfrak{g}^k$ by definition. So, it remains to prove that for all x in $\mathcal{C}^{(k)}$, $\eta^{-1}(x)$ is contained in $\overline{G.\iota_k(\mathfrak{h}^k)}$. There is a canonical action of $\mathrm{GL}_k(\Bbbk)$ on $\mathfrak{g}^k$ and $\mathcal{X}^k$. Since this action commutes with the action of G in $\mathcal{X}^k$, $\mathcal{B}^{(k)}_{\rm x}$ is invariant under $\mathrm{GL}_k(\Bbbk)$ and η is $\mathrm{GL}_k(\Bbbk)$-equivariant. As a result, since $\mathcal{C}^{(k)}$ and $G.\iota_k(\mathfrak{h}^k)$ are invariant under $\mathrm{GL}_k(\Bbbk)$, for x in $\mathcal{C}^{(k)}$, $\eta^{-1}(x')$ is contained in $\overline{G.\iota_k(\mathfrak{h}^k)}$ for all x' in E^k_x such that $E_{x'} = E_x$ if $\eta^{-1}(x)$ is contained in $\overline{G.\iota_k(\mathfrak{h}^k)}$. Then, according to Lemma 5.2, since η is G-equivariant, it suffices to prove that $\eta^{-1}(x)$ is contained in $\overline{G.\iota_k(\mathfrak{h}^k)}$ for x in $\mathcal{C}^{(k)} \cap \mathfrak{b}^k$ verifying Conditions (1) and (2) of Lemma 5.2 for some s in $\mathfrak{h}$.

According to Corollary 5.5,

$$\eta^{-1}(x) \subset \{(x_1, \dots, x_k, w(x_{1,\rm s}), \dots, w(x_{k,\rm s})) \mid w \in W(\mathcal{R})\} \text{ with } x = (x_1, \dots, x_k).$$

For s regular, E_x is contained in $\mathfrak{h}$ and $x_i = x_{i,\rm s}$ for $i = 1, \dots, k$. By definition,

$$(w(x_1), \dots, w(x_k), w(x_1), \dots, w(x_k)) \in \iota_k(\mathfrak{h}^k)$$

and for g_w a representative of w in $N_G(\mathfrak{h})$,

$$g_w^{-1}.(w(x_1), \dots, w(x_k), w(x_1), \dots, w(x_k)) = (x_1, \dots, x_k, w(x_1), \dots, w(x_k)).$$

Hence $\eta^{-1}(x)$ is contained in $G.\iota_k(\mathfrak{h}^k)$. As a result, according to the notations of Lemma 5.1, for all w in $W(\mathcal{R})$, $\theta_w(\mathcal{E}_0^{(k,*)})$ is contained in $G.\iota_k(\mathfrak{h}^k)$. Hence, by Lemma 5.1,(ii), $\theta_w(\mathcal{E}_0^{(k)})$ is contained in $\overline{G.\iota_k(\mathfrak{h}^k)}$, whence the proposition. □

5.3. Let ϖ be the canonical projection from $\mathcal{X}^k$ to $\mathfrak{g}^k$. By Corollary 2.6,(ii), $\mathcal{B}_{\mathrm{x}}^{(k)}$ is an irreducible component of $\varpi^{-1}(\mathcal{B}^{(k)})$ and the action of $W(\mathcal{R})^k$ on $\mathcal{X}^k$ induces a simply transitive action on the set of irreducible components of $\varpi^{-1}(\mathcal{B}^{(k)})$. According to Remark 2.21, there is an embedding Φ of $\mathrm{S}(\mathfrak{h})^{\otimes k}$ into $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]$ given by

$$p \longmapsto ((x_1,\dots,x_k,y_1,\dots,y_k) \mapsto p(y_1,\dots,y_k)).$$

By Corollary 2.22,(i), this embedding identifies $\mathrm{S}(\mathfrak{h})^{\otimes k}$ with $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]^G$.

Lemma 5.7. *Let Ψ be the restriction to $\mathrm{S}(\mathfrak{h})^{\otimes k}$ of the canonical map from $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]$ to $\Bbbk[\mathcal{C}_{\mathrm{x}}^{(k)}]$.*

(i) *The subvariety $\mathcal{C}_{\mathrm{x}}^{(k)}$ of $\mathcal{X}^k$ is invariant under the diagonal action of $W(\mathcal{R})$ in $\mathcal{X}^k$.*

(ii) *The map Ψ is an embedding of $\mathrm{S}(\mathfrak{h})^{\otimes k}$ into $\Bbbk[\mathcal{C}_{\mathrm{x}}^{(k)}]$. Moreover, $\Psi(\mathrm{S}(\mathfrak{h})^{\otimes k})$ is equal to $\Bbbk[\mathcal{C}_{\mathrm{x}}^{(k)}]^G$.*

(iii) *The image of $(\mathrm{S}(\mathfrak{h})^{\otimes k})^{W(\mathcal{R})}$ by Ψ equals $\Bbbk[\mathcal{C}^{(k)}]^G$.*

Proof. (i) For x in $\mathcal{B}_{\mathrm{x}}^{(k)}$ and w in $W(\mathcal{R})$, $\eta(x) = \eta(w.x)$, whence the assertion by Proposition 5.6.

(ii) For P in $\mathrm{S}(\mathfrak{h})^{\otimes k}$, $P = 0$ if $P(x) = 0$ for all x in $\iota_k(\mathfrak{h}^k)$. Hence Ψ is injective. Since G is reductive, $\Bbbk[\mathcal{C}_{\mathrm{x}}^{(k)}]^G$ is the image of $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]^G$ by the quotient morphism, whence the assertion.

(iii) Since G is reductive, $\Bbbk[\mathcal{C}^{(k)}]^G$ is the image of $\Bbbk[\mathcal{B}^{(k)}]^G$ by the quotient morphism, whence the assertion since $(\mathrm{S}(\mathfrak{h})^{\otimes k})^{W(\mathcal{R})}$ is equal to $\Bbbk[\mathcal{B}^{(k)}]^G$ by Corollary 2.22,(iii). □

Identify $\mathrm{S}(\mathfrak{h})^{\otimes k}$ with $\Bbbk[\mathcal{C}_{\mathrm{x}}^{(k)}]^G$ by Ψ.

Proposition 5.8. *Let $\widetilde{\mathcal{C}_{\mathrm{x}}^{(k)}}$ and $\widetilde{\mathcal{C}^{(k)}}$ be the normalizations of $\mathcal{C}_{\mathrm{x}}^{(k)}$ and $\mathcal{C}^{(k)}$.*

(i) *The variety $\mathcal{C}^{(k)}$ is the categorical quotient of $\mathcal{C}_{\mathrm{x}}^{(k)}$ under the action of $W(\mathcal{R})$.*

(ii) *The variety $\widetilde{\mathcal{C}^{(k)}}$ is the categorical quotient of $\widetilde{\mathcal{C}_{\mathrm{x}}^{(k)}}$ under the action of $W(\mathcal{R})$.*

Proof. (i) According to Corollary 2.22,(i), $\Bbbk[\mathcal{B}_{\mathrm{x}}^{(k)}]$ is generated by $\Bbbk[\mathcal{B}^{(k)}]$ and $\mathrm{S}(\mathfrak{h})^{\otimes k}$. Since $\mathcal{C}_{\mathrm{x}}^{(k)} = \eta^{-1}(\mathcal{C}^{(k)})$ by Proposition 5.6, the image of $\Bbbk[\mathcal{B}^{(k)}]$ in $\Bbbk[\mathcal{C}_{\mathrm{x}}^{(k)}]$ by the quotient morphism is equal to $\Bbbk[\mathcal{C}^{(k)}]$. Hence $\Bbbk[\mathcal{C}_{\mathrm{x}}^{(k)}]$ is generated by $\Bbbk[\mathcal{C}^{(k)}]$ and $\mathrm{S}(\mathfrak{h})^{\otimes k}$. Then, by Lemma 5.7,(iii), $\Bbbk[\mathcal{C}_{\mathrm{x}}^{(k)}]^{W(\mathcal{R})} = \Bbbk[\mathcal{C}^{(k)}]$.

(ii) Let K be the fraction field of $\Bbbk[\mathcal{C}_{\mathrm{x}}^{(k)}]$. Since $\mathcal{C}_{\mathrm{x}}^{(k)}$ is a $W(\mathcal{R})$-variety, there is an action of $W(\mathcal{R})$ in K and $K^{W(\mathcal{R})}$ is the fraction field of $\Bbbk[\mathcal{C}_{\mathrm{x}}^{(k)}]^{W(\mathcal{R})}$ since $W(\mathcal{R})$ is finite. As a result, the integral closure $\Bbbk[\widetilde{\mathcal{C}_{\mathrm{x}}^{(k)}}]$ of $\Bbbk[\mathcal{C}_{\mathrm{x}}^{(k)}]$ in K is invariant under $W(\mathcal{R})$ and $\Bbbk[\widetilde{\mathcal{C}^{(k)}}]$ is contained in $\Bbbk[\widetilde{\mathcal{C}_{\mathrm{x}}^{(k)}}]^{W(\mathcal{R})}$ by (i). Let a be in $\Bbbk[\widetilde{\mathcal{C}_{\mathrm{x}}^{(k)}}]^{W(\mathcal{R})}$. Then a verifies a dependence integral equation over $\Bbbk[\mathcal{C}_{\mathrm{x}}^{(k)}]$,

$$a^m + a_{m-1}a^{m-1} + \cdots + a_0 = 0$$

whence

$$a^m + \Big(\frac{1}{|W(\mathcal{R})|}\sum_{w\in W(\mathcal{R})} w.a_{m-1}\Big)a^{m-1} + \cdots + \frac{1}{|W(\mathcal{R})|}\sum_{w\in W(\mathcal{R})} w.a_0 = 0$$

since a is invariant under $W(\mathcal{R})$ so that a is in $\Bbbk[\widetilde{\mathcal{C}^{(k)}}]$ by (i), whence the assertion. □

6. Desingularization

Let $k \geq 2$ be an integer. Let X, X' be as in Subsection 4.5. Denote by X_{n} the normalization of X and by θ_0 the normalization morphism. According to Proposition 4.15, X' identifies with a smooth big open subset of X_{n} and according to [Hir64], there exists a desingularization $(\Gamma, \pi_{\mathrm{n}})$ of X_{n} in the category of B-varieties such that the restriction of π_{n} to $\pi_{\mathrm{n}}^{-1}(X')$ is an isomorphism onto X'. Set $\pi = \theta_0 \circ \pi_{\mathrm{n}}$ so that (Γ, π) is a desingularization of X in the category of B-varieties. Recall that $\mathcal{E}_0$ is the restriction to X of the tautological vector bundle over $\mathrm{Gr}_\ell(\mathfrak{g})$ and $\mathfrak{X}_{0,k}$ is the closure in $\mathfrak{b}^k$ of $B.\mathfrak{h}^k$. Set $\mathfrak{X}_k := G \times_B \mathfrak{X}_{0,k}$. Then $\mathfrak{X}_k$ is a closed subvariety of $G \times_B \mathfrak{b}^k$.

Lemma 6.1. *Let τ' be the canonical morphism from $\mathcal{E}_0$ to $\mathfrak{b}$.*

(i) *The morphism τ' is projective and birational.*

(ii) *Let ν be the canonical map from $\pi^*(\mathcal{E}_0)$ to $\mathcal{E}_0$. Then ν and $\tau := \tau' \circ \nu$ are B-equivariant birational projective morphisms from $\pi^*(\mathcal{E}_0)$ to $\mathcal{E}_0$ and $\mathfrak{b}$ respectively. In particular, $\pi^*(\mathcal{E}_0)$ is a desingularization of $\mathcal{E}_0$ and $\mathfrak{b}$.*

Proof. (i) Since X is a projective variety, τ' is a projective morphism and $\tau'(\mathcal{E}_0)$ is closed in $\mathfrak{b}$. Moreover, $\tau'(\mathcal{E}_0)$ is B-invariant since τ' is a B-equivariant morphism and it contains $\mathfrak{h}$ since $\mathfrak{h}$ is in X. For x in $\mathfrak{h}_{\mathrm{reg}}$, $(\tau')^{-1}(x) = \{(\mathfrak{h}, x)\}$. Hence τ' is a birational morphism and $\tau'(\mathcal{E}_0) = \mathfrak{b}$ since $B(\mathfrak{h}_{\mathrm{reg}})$ is an open subset of $\mathfrak{b}$.

(ii) Since $\mathcal{E}_0$ is a vector bundle over X and since π is a projective birational morphism, ν is a projective birational morphism. Then τ is a projective birational morphism from $\pi^*(\mathcal{E}_0)$ to $\mathfrak{b}$ by (i). It is B-equivariant since so are ν and τ'. Moreover, $\pi^*(\mathcal{E}_0)$ is a desingularization of $\mathcal{E}_0$ and $\mathfrak{b}$ since $\pi^*(\mathcal{E}_0)$ is smooth as a vector bundle over a smooth variety. □

Denote by ψ the canonical projection from $\pi^*(\mathcal{E}_0)$ to Γ. Then, according to the above notations, we have the commutative diagram:

$$\begin{tikzcd}
 & \pi^*(\mathcal{E}_0) \arrow[r, "\psi"] \arrow[dl, "\tau"'] \arrow[d, "\nu"'] & \Gamma \arrow[d, "\pi"] \\
\mathfrak{b} & \mathcal{E}_0 \arrow[l, "\tau'"] \arrow[r] & X
\end{tikzcd}$$

Recall that $\mathcal{E}_0^{(k)}$ is the subvariety of $X \times \mathfrak{b}^k$:

$$\mathcal{E}_0^{(k)} := \{(u, x_1, \ldots, x_k) \in X \times \mathfrak{b}^k \mid u \ni x_1, \ldots, u \ni x_k\}.$$

As $\mathcal{E}_0$ is a vector bundle over X, so is $\mathcal{E}_0^{(k)}$.

Lemma 6.2. *Set $\mathcal{E}_{\mathrm{s}}^{(k)} := \pi^*(\mathcal{E}_0^{(k)})$. Let τ_k be the canonical morphism from $\mathcal{E}_{\mathrm{s}}^{(k)}$ to $\mathfrak{b}^k$.*

(i) *The vector bundle $\mathcal{E}_{\mathrm{s}}^{(k)}$ over Γ is a vector subbundle of the trivial bundle $\Gamma \times \mathfrak{b}^k$. Moreover, $\mathcal{E}_{\mathrm{s}}^{(k)}$ has dimension $k\ell + n$.*

(ii) *The morphism τ_k is a projective birational morphism from $\mathcal{E}_{\mathrm{s}}^{(k)}$ onto $\mathfrak{X}_{0,k}$. Moreover, $\mathcal{E}_{\mathrm{s}}^{(k)}$ is a desingularization of $\mathfrak{X}_{0,k}$ in the category of B-varieties.*

Proof. (i) By definition, $\mathcal{E}_{\mathrm{s}}^{(k)}$ is the subvariety of $\Gamma \times \mathfrak{b}^k$. Since X and Γ have dimension n, $\mathcal{E}_{\mathrm{s}}^{(k)}$ has dimension $k\ell + n$ as a vector bundle of rank $k\ell$ over Γ.

(ii) Since Γ is a projective variety, τ_k is a projective morphism and $\tau_k(\mathcal{E}_{\mathrm{s}}^{(k)}) = \mathfrak{X}_{0,k}$ by Lemma 5.1,(i). For $(x_1, \ldots, x_k)$ in $\mathfrak{b}^k_{\mathrm{reg,ss}} \cap \mathfrak{X}_{0,k}$, $\tau_k^{-1}(x_1, \ldots, x_k) = \{(\pi^{-1}(\mathfrak{g}^{x_1}), (x_1, \ldots, x_k))\}$ since

$\mathfrak{g}^{x_1}$ is a Cartan subalgebra. Hence τ_k is a birational morphism, whence the assertion since $\mathcal{E}_{\mathrm{s}}^{(k)}$ is a smooth B-variety as a vector bundle over the smooth B-variety Γ. □

Set $\mathfrak{Y} := G \times_B (\Gamma \times \mathfrak{b}^k)$. The canonical projections from $G \times \Gamma \times \mathfrak{b}^k$ to $G \times \Gamma$ and $G \times \mathfrak{b}^k$ define through the quotients morphisms from $\mathfrak{Y}$ to $G \times_B \Gamma$ and $G \times_B \mathfrak{b}^k$. Denote by ς and ζ these morphisms. Then we have the following diagram:

$$\begin{tikzcd} \mathfrak{Y} \arrow[r, "\zeta"] \arrow[d, "\varsigma"'] & G \times_B \mathfrak{b}^k \arrow[d, "\gamma_{\mathrm{x}}"] \\ G \times_B \Gamma & \mathcal{B}_{\mathrm{x}}^{(k)} \end{tikzcd}$$

The map $(g, x) \mapsto (g, \tau_k(x))$ from $G \times \mathcal{E}_{\mathrm{s}}^{(k)}$ to $G \times \mathfrak{b}^k$ defines through the quotient a morphism $\overline{\tau_k}$ from $G \times_B \mathcal{E}_{\mathrm{s}}^{(k)}$ to $\mathfrak{X}_k$.

Proposition 6.3. *Set* $\xi := \gamma_{\mathrm{x}} \circ \overline{\tau_k}$.

(i) *The variety* $G \times_B \mathcal{E}_{\mathrm{s}}^{(k)}$ *is a closed subvariety of* $\mathfrak{Y}$.

(ii) *The variety* $G \times_B \mathcal{E}_{\mathrm{s}}^{(k)}$ *is a vector bundle of rank* $k\ell$ *over* $G \times_B \Gamma$. *Moreover,* $G \times_B \Gamma$ *and* $G \times_B \mathcal{E}_{\mathrm{s}}^{(k)}$ *are smooth varieties.*

(iii) *The morphism* ξ *is a projective birational morphism from* $G \times_B \mathcal{E}_{\mathrm{s}}^{(k)}$ *onto* $\mathcal{C}_{\mathrm{x}}^{(k)}$.

Proof. (i) According to Lemma 6.2,(i), $\mathcal{E}_{\mathrm{s}}^{(k)}$ is a closed subvariety of $\Gamma \times \mathfrak{b}^k$, invariant under the diagonal action of B. Hence $G \times \mathcal{E}_{\mathrm{s}}^{(k)}$ is a closed subvariety of $G \times \Gamma \times \mathfrak{b}^k$, invariant under the action of B, whence the assertion.

(ii) Since $\mathcal{E}_{\mathrm{s}}^{(k)}$ is a B-equivariant vector bundle over Γ, $G \times_B \mathcal{E}_{\mathrm{s}}^{(k)}$ is a G-equivariant vector bundle over $G \times_B \Gamma$. Since $G \times_B \Gamma$ is a fiber bundle over the smooth variety G/B with smooth fibers, $G \times_B \Gamma$ is a smooth variety. As a result, $G \times_B \mathcal{E}_{\mathrm{s}}^{(k)}$ is a smooth variety.

(iii) According to Lemma 6.2,(ii) and Lemma 1.7, $\overline{\tau_k}$ is a projective birational morphism from $G \times_B \mathcal{E}_{\mathrm{s}}^{(k)}$ to $\mathfrak{X}_k$. Since $\mathfrak{X}_{0,k}$ is a B-invariant closed subvariety of $\mathfrak{b}^k$, $\mathfrak{X}_k$ is closed in $G \times_B \mathfrak{b}^k$. According to Lemma 5.1,(i), $\gamma(\mathfrak{X}_k) = \mathcal{C}^{(k)}$. Moreover, $\gamma_{\mathrm{x}}(\mathfrak{X}_k)$ is an irreducible closed subvariety of $\mathcal{B}_{\mathrm{x}}^{(k)}$ since γ_{x} is a projective morphism by Lemma 1.7. Hence $\gamma_{\mathrm{x}}(\mathfrak{X}_k) = \mathcal{C}_{\mathrm{x}}^{(k)}$ by Proposition 5.6. For all z in $G.\iota_k(\mathfrak{h}_{\mathrm{reg}}^k)$, $|\gamma_{\mathrm{x}}^{-1}(z)| = 1$. Hence the restriction of γ_{x} to $\mathfrak{X}_k$ is a birational morphism onto $\mathcal{C}_{\mathrm{x}}^{(k)}$ since $G.\iota_k(\mathfrak{h}_{\mathrm{reg}}^k)$ is dense in $\mathcal{C}_{\mathrm{x}}^{(k)}$. Moreover, this morphism is projective since γ_{x} is projective. As a result, ξ is a projective birational morphism from $G \times_B \mathcal{E}_{\mathrm{s}}^{(k)}$ onto $\mathcal{C}_{\mathrm{x}}^{(k)}$. □

Theorem 1.3 results from Proposition 5.6 and Proposition 6.3,(ii) and (iii) and the following corollary results from Lemma 6.2,(ii), Proposition 6.3,(ii) and (iii), and Lemma 1.4.

Corollary 6.4. *Let* $\widetilde{\mathfrak{X}_{0,k}}$ *and* $\widetilde{\mathcal{C}_{\mathrm{x}}^{(k)}}$ *be the normalizations of* $\mathfrak{X}_{0,k}$ *and* $\mathcal{C}_{\mathrm{x}}^{(k)}$ *respectively. Then* $\Bbbk[\widetilde{\mathfrak{X}_{0,k}}]$ *and* $\Bbbk[\widetilde{\mathcal{C}_{\mathrm{x}}^{(k)}}]$ *are the spaces of global sections of* $\mathcal{O}_{\mathcal{E}_{\mathrm{s}}^{(k)}}$ *and* $\mathcal{O}_{G \times_B \mathcal{E}_{\mathrm{s}}^{(k)}}$ *respectively.*

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